b-divisors on toric and toroidal embeddings

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Abstract

In this thesis we develop an intersection theory of toric and toroidal b-divisors on toric and toroidal embeddings, respectively. Our motivation comes from wanting to establish an arithmetic intersection theory on mixed Shimura varieties of non-compact type. The tools available until now do not define numerical invariants which are birationally invariant.

First, we define toric b-divisors on toric varieties and an integrability notion of such divisors. We show that under suitable positivity assumptions toric b-divisors are integrable and that their degree is given as the volume of a convex set. Moreover, we show that the dimension of the space of global sections of a nef toric b-divisor is equal to the number of lattice points in this convex set and we give a Hilbert–Samuel type formula for its asymptotic growth. This generalizes classical results for classical toric divisors on toric varieties. As a by-product, we relate convex sets arising from toric b-divisors with Newton–Okounkov bodies.

Then, we define toroidal b-divisors on toroidal varieties and an integrability notion of such divisors. We show that under suitable positivity assumptions toroidal b-divisors are integrable and that their degree is given as an integral with respect to a limit measure, which is a weak limit of discrete measures whose weights are defined via tropical intersection theory on the rational conical polyhedral complex attached to the toroidal variety. We also relate this limit measure with the surface area measure associated to a convex body. This relation enables us to compute integrals with respect to these limit measures explicitly. Additionally, we give a canonical decomposition of the difference of two convex sets and we relate the volumes of the pieces with tropical top intersection numbers.

Finally, as an application, we compute the degree of the b-divisor of Jacobi forms of weight k and index m with respect to the principal congruence subgroup of level $N \geq 3$ on the generalized universal elliptic curve and we show that it is meaningful to consider the b-divisorial approach instead of just fixing one canonical compactification.

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Chapter 0

Introduction

0.1 Background

One of the main goals of arithmetic geometry is to be able to “measure” the arithmetic complexity of an arithmetic object. This has led to the development of the theory of heights, numerical invariants defined via the theory of arithmetic intersections. Arithmetic intersection theory, which is also known as Arakelov theory, is a way to study varieties over rings of integers of number fields by putting smooth hermitian metrics on holomorphic vector bundles over the complex points of the variety. The foundations of this theory go back to Arakelov in [Ara74, Ara75], where he developed the theory for arithmetic surfaces. This theory was generalized to higher dimensional arithmetic varieties by Gillet–Soulé in [GS92]. Later on, Burgos–Kramer–Kühn were able to weaken the smoothness condition of the metrics so as to allow so called log-singular metrics ([BKK05] and [BKK07]). The key result used in order to develop the theory with this kind of singular metrics was the following theorem by Mumford [Mum77].

Theorem 0.1.1. *Every automorphic line bundle on a pure open Shimura variety, equipped with an invariant smooth metric, can be uniquely extended as a line bundle on a toroidal compactification of the variety, in such a way that the metric acquires only logarithmic singularities.*

Applications of arithmetic intersection theory with log-singular metrics have been found in the study of arithmetic cycles in pure Shimura varieties of non-compact type such as in Kudla’s programme. The latter seeks to relate generating series of intersection numbers of arithmetic cycles to the Fourier coefficients of modular forms.

A natural question is whether Mumford’s theorem is valid also in the mixed Shimura setting. As was noticed in [BKK16], it is no longer valid in this situation: the invariant metric on an automorphic line bundle over a mixed Shimura variety of non-compact type acquires worse than logarithmic singularities over a toroidal compactification. As a consequence of the appearance of this new type of singularities, a “naive” extension of the metric along the boundary is no good as it depends on the choice of a compactification and does not satisfy Chern–Weil theory.

This thesis is part of a general goal to propose a method of dealing with these new singularity types via convex geometric methods in the case of toroidal compactifications. The idea is to take *all* possible toroidal compactifications into account and encode the singularity type in a so called toroidal b-divisor (Definition 4.2.5). The correction terms needed to define a meaningful intersection theory should correspond to mixed degrees of such b-divisors (Definition 4.4.4).

In this thesis we address the geometric side of the problem and we develop an intersection theory of b-divisors on toric and toroidal varieties. This serves as a starting point to develop an arithmetic intersection theory on mixed Shimura varieties of non-compact type.

0.2 Objects of study

We work over an algebraically closed field k of characteristic 0. For the sake of simplicity, we illustrate the theory in the toric case. Let N be a lattice of rank n. We fix a complete, smooth fan $\Sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and we let X_Σ be the corresponding n-dimensional complete, smooth toric variety with dense open torus $\mathbb{T}$. The set $R(\Sigma)$ consists of all smooth sudivisions of Σ. This is a directed set with partial order given by $\Sigma'' \geq \Sigma'$ in $R(\Sigma)$ if and only if Σ'' is a smooth subdivision of Σ'. The *toric Riemann–Zariski space* of X_Σ is defined as the inverse limit

$$\mathfrak{X}_\Sigma := \varprojlim_{\Sigma' \in R(\Sigma)} X_{\Sigma'},$$

with maps given by the toric proper birational morphisms $\pi_{\Sigma''}\colon X_{\Sigma''} \to X_{\Sigma'}$ induced whenever $\Sigma'' \geq \Sigma'$. We aim to study the group of *toric Weil b-divisors* on X_Σ, which are elements in the inverse limit

$$\mathrm{We}(\mathfrak{X}_\Sigma)_{\mathbb{Q}} := \varprojlim_{\Sigma' \in R(\Sigma)} \mathbb{T}\text{-}\mathrm{Ca}(X_{\Sigma'})_{\mathbb{Q}},$$

where $\mathbb{T}\text{-}\mathrm{Ca}(X_{\Sigma'})_{\mathbb{Q}}$ denotes the set of toric Cartier $\mathbb{Q}$-divisors of $X_{\Sigma'}$, with maps given by the push-forward map of toric Cartier $\mathbb{Q}$-divisors (Definition 1.2.3). We will denote b-divisors in bold $\boldsymbol{D}$ to distinguish them from classical divisors D. We can think of a toric b-divisor as a net of toric Cartier $\mathbb{Q}$-divisors $(D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$, being compatible under push-forward.

A toric b-divisor $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ is said to be *nef*, if $D_{\Sigma'} \in \mathbb{T}\text{-}\mathrm{Ca}(X_{\Sigma'})_{\mathbb{Q}}$ is nef for all Σ' in a cofinal subset of $R(\Sigma)$. It follows from basic toric geometry that there is a bijective correspondence between the set of nef toric b-divisors and the set of $\mathbb{Q}$-valued, conical, $\mathbb{Q}$-concave functions on $N_{\mathbb{Q}}$.

The *mixed degree* $\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$ of a collection of toric b-divisors is defined as the limit (in the sense of nets)

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n := \lim_{\Sigma' \in R(\Sigma)} D_{1_{\Sigma'}} \cdots D_{n_{\Sigma'}}$$

of top intersection numbers of toric divisors, provided this limit exists and is finite. In particular, if $\boldsymbol{D} = \boldsymbol{D}_1 = \ldots = \boldsymbol{D}_n$, then the limit (in the sense of nets)

$$\boldsymbol{D}^n := \lim_{\Sigma' \in R(\Sigma)} D^n_{\Sigma'},$$

provided this limit exists and is finite, is called the *degree* of the toric b-divisor $\boldsymbol{D}$. A toric b-divisor whose degree exists, is said to be *integrable*.

0.3 Main results of this thesis

Toric case

Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ be a collection of toric b-divisors on a smooth and complete toric variety X_Σ of dimension n which are nef. Let $\tilde{\phi}_i\colon N_{\mathbb{Q}} \to \mathbb{Q}$ be the corresponding $\mathbb{Q}$-concave functions for $i = 1, \ldots, n$.

Theorem 1.3.10/1.3.13. *With notations as above, the functions $\tilde{\phi}_i$ extend to continuous, concave functions $\phi_i\colon N_{\mathbb{R}} \to \mathbb{R}$. Moreover, their mixed degree $\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$ exists, and is given by the mixed volume of the stability sets Δ_{ϕ_i} of the concave functions ϕ_i, i.e. we have that*

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n = \mathrm{MV}\left(\Delta_{\phi_1}, \dots, \Delta_{\phi_n}\right).$$

In particular, a nef toric b-divisor $\boldsymbol{D}$ is integrable, and its degree is given by

$$\boldsymbol{D}^n = n!\,\mathrm{vol}\left(\Delta_{\phi}\right),$$

where ϕ is the corresponding concave function.

We can also define the space of global sections $H^0(X_\Sigma, \boldsymbol{D})$ of a toric b-divisor $\boldsymbol{D}$ (Definition 1.5.6). The following result also generalizes the classical toric setting.

Proposition 1.5.10/Theorem 1.5.11. *Let $\boldsymbol{D}$ be a toric b-divisor. There is a canonically defined convex set $\Delta_{\boldsymbol{D}}$ inducing an isomorphism*

$$H^0(X_\Sigma, \boldsymbol{D}) \simeq \bigoplus_{m \in M \cap \Delta_{\boldsymbol{D}}} k \cdot \chi^m,$$

where χ^m is the torus character associated to the lattice point m. Moreover, if $\boldsymbol{D}$ is nef and if ϕ is its associated concave function on $N_{\mathbb{R}}$ then $\Delta_{\boldsymbol{D}} = \Delta_{\phi}$ and the following Hilbert–Samuel type formula holds true

$$\boldsymbol{D}^n = \lim_{\ell \to \infty} \frac{h^0(X_\Sigma, \ell\boldsymbol{D})}{\ell^n / n!}.$$

The concept of a toric b-divisor given in this thesis appears to be new as well as the theory of toric b-divisors developed here.

Relationship with Okounkov bodies

Okounkov bodies are convex bodies which one can attach to an algebraic variety together with some extra data, e.g. a complete flag of subvarieties. These convex bodies have been widely and successfully used to explore the geometry of the variety using convex geometrical methods ([Oko96, Oko03, KK12, KK14, KK08, LM09]). In [KK12], Kaveh and Kovanskii attached an Okounkov body $\Delta_A \subseteq \mathbb{R}^n$ to a so called algebra of almost integral type A (Definition 2.1.2). It turns out that the ring of global sections of multiples of a toric b-divisor $\boldsymbol{D}$ defines such an algebra of almost integral type $A_{\boldsymbol{D}}$. We identify the convex bodies $\Delta_{A_{\boldsymbol{D}}}$ and $\Delta_{\boldsymbol{D}}$.

Proposition 2.1.13/Theorem 2.1.15. *Let $\boldsymbol{D}$ be a toric b-divisor on X_Σ. Then the graded algebra*

$$A_{\boldsymbol{D}} = \bigoplus_{\ell \geq 0} H^0(X_\Sigma, \ell\boldsymbol{D}) t^\ell$$

is an algebra of almost integral type. Moreover, there is an identification $M \simeq \mathbb{R}^n$ such that

$$\Delta_{A_{\boldsymbol{D}}} = \Delta_{\boldsymbol{D}}$$

under this identification.

As an application of this identification, in the big and nef case, we construct a global b-convex body $\Delta(\mathfrak{X}_\Sigma)$ which lives over the topological space of numerical classes of toric b-divisors on X_Σ, whose fibre over any big and nef class $[\boldsymbol{D}]$ is the corresponding convex body $\Delta_{\boldsymbol{D}}$ (Theorem 2.2.12). This construction generalizes the global Okounkov body associated to a big divisor constructed by Lazarsfeld and Mustaţă in [LM09] to the b-setting and gives an alternative insight into the birational geometry of toric varieties.

The relationship between Okounkov bodies and polytopes coming from the classical theory of toric varieties is well established and appears throughout the literature (see e.g [LM09]). The generalization to a relationship between Okounkov bodies and convex sets arising from toric b-divisors seems to be new.

Toroidal case

Let $U \hookrightarrow X$ be a smooth toroidal embedding of dimension n with associated (weakly embedded) smooth conical rational polyhedral complex Π_X (Definitions 3.1.1 and 3.1.13). The set $\operatorname{Div}_0(X)$ denotes the set of toroidal divisors on X, i.e. supported on the boundary $X \setminus U$. Generalizing the toric situation, we define the directed set $R(\Pi_X)$ consisting of all smooth subdivisions of Π_X. An element Π' in $R(\Pi_X)$ corresponds to a smooth toroidal embedding $X_{\Pi'}$ together with a toroidal, proper birational morphism $\pi_{\Pi'} \colon X_{\Pi'} \to X$ (Theorem 4.1.23). We define the group of *toroidal Weil b-divisors* as elements in the inverse limit

$$\operatorname{We}(\mathfrak{X}_{\Pi_X})_{\mathbb{Q}} := \varprojlim_{\Pi' \in R(\Pi_X)} \operatorname{Div}_0(X_{\Pi'})_{\mathbb{Q}},$$

with maps given by the push-forward map of toroidal Cartier divisors (Definition 4.2.5). A nef toroidal b-divisor $\boldsymbol{D}$ induces a continuous function $\phi_{\boldsymbol{D}}$ on the support of the conical complex $|\Pi_X|$ whose restriction to each cone $\sigma \in \Pi_X$ is concave (Theorem 4.4.3). We have the following result.

Theorem 4.4.5. *The mixed degree of a collection of nef toroidal b-divisors $\boldsymbol{D}_1, \dots, \boldsymbol{D}_n$ on X, defined as the limit (in the sense of nets)*

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n := \lim_{\Pi' \in R(\Pi_X)} D_{1_{\Pi'}} \cdots D_{n_{\Pi'}},$$

exists, is finite, and is given by

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n = \int_{\mathbb{S}^{\Pi}} \phi_{\boldsymbol{D}_1}(u) \mu_{\phi_{\boldsymbol{D}_2}, \dots, \phi_{\boldsymbol{D}_n}},$$

where $\mu_{\phi_{\boldsymbol{D}_2}, \dots, \phi_{\boldsymbol{D}_n}}$ is a mixed limit measure supported on a compact subset $\mathbb{S}^{\Pi_X} \subseteq |\Pi_X|$, obtained as a weak limit of discrete measures supported on the rays of the polyhedral complexes $\Pi' \in R(\Pi_X)$ with weights defined via tropical intersection theory.

One of the key ingredients to prove the above theorem is the combinatorial machinery developed in Chapter 3. The existence of the limit measure $\mu_{\phi_{\boldsymbol{D}}}$ associated to a nef toroidal b-divisor $\boldsymbol{D}$ follows directly from Corollary 3.3.25 and the existence of the mixed limit measure $\mu_{\phi_{\boldsymbol{D}_2}, \dots, \phi_{\boldsymbol{D}_n}}$ is a direct consequence of Corollary 3.3.28. Another key ingredient is a result in [Gro15] relating tropical and algebraic intersection numbers (Theorem 4.3.4).

Moreover, we show that the limit measure $\mu_{\phi_{\boldsymbol{D}}}$ and the mixed limit measure $\mu_{\phi_{\boldsymbol{D}_2}, \dots, \phi_{\boldsymbol{D}_n}}$ generalize the so called surface area measure associated to a convex body and the mixed version thereof ([Sch93]) to the rational conical polyhedral complex case (Section 3.4). This enables us to compute top intersection numbers of nef toroidal b-divisors explicitly in some cases in terms of Lebesgue integrals of determinants of Hessians of smooth functions (Corollary 3.4.18 and Proposition 3.4.22).

As a by-product of the combinatorial results given in the first sections of Chapter 3, in Section 3.5 we give a canonical decomposition of the complement of two convex sets and we interpret the volumes of the pieces in terms of tropical top intersection numbers (Proposition 3.5.21 and Theorem 3.5.27). In the polyhedral case, the above canonical decomposition gives a polyhedral subdivision of the complement of two polytopes, one contained in the other. This subdivision appears in the literature (e.g. in [GP88]) although it is constructed using the so called pushing method. We haven't found in the literature the method we used in Proposition 3.5.21 nor have we found such a canonical decomposition in the non-polyhedral case.

The concept of a toroidal b-divisor given here seems to be new as well as the theory of toroidal b-divisors developed in this thesis. As for Chapter 3, the definition of the discrete measures of Definition 3.2.23 seems to be new as well as the method of proof for the weak convergence given in Section 3.3. Also the concept of b-divisors on a conical complex (Definition 3.3.4) seems to be given in this thesis for the first time.

Applications

As an application of the intersection theory of b-divisors on toroidal varieties developed in Chapter 4, in Chapter 5 we compute the degree of the b-divisor of Jacobi forms of weight k and index m with respect to the principal congruence subgroup $\Gamma(N) \subseteq \mathrm{PSL}_2(\mathbb{Z})$ of level N. We also show why it is meaningful to consider the b-divisorial approach instead of just fixing one canonical compatification. On the one hand side, this is done by giving a geometric interpretation of the space of global sections.

Theorem 5.3.3. *There is an isomorphism between the space of global sections of the b-divisor of Jacobi forms and the space of Jacobi cusp forms.*

The idea that the canonical object to consider when wanting to describe the space Jacobi cusp forms should be a limit was already indicated by Kramer in [Kra95, Remark 2.19]. On the other hand, we give a Hilbert–Samuel type formula relating the degree of the b-divisor of Jacobi forms and the asymptotic growth of the space of global sections of multiples of the b-divisor (Corollary 5.3.5) and we show that Chern–Weil theory holds in this context (Theorem 5.3.7). The latter two results were shown in [BKK16, Theorem 5.1, Theorem 5.2] for the case $k = m = 4$. We also give the definition of a b-line bundle (Definition 5.2.2), a concept which did not appear in [BKK16]. This definition is taken from a (still unpublished) article by M. Jespers and R. de Jong. Chapter 5 can be seen as a generalization and a conceptual interpretation of the results given in [BKK16].

We remark that we have computed explicitly the toroidal b-divisor encoding the singularity type of the invariant metric on an automorphic line bundle also in the following (mixed Shimura) cases:

- The self fibre product of the universal elliptic curve with a level structure together with the Poincaré bundle.
- The principally polarized universal abelian surface with level structure together with the theta line bundle.

These results however are not included in this thesis and will form part of a future work.

Others

In Appendix A we give a possible starting point to study integrability questions of (not necessarily nef) toric *b*-divisors in smooth and complete toric varieties which uses a formula for the degree of (not necessarily nef) toric divisors in terms of volumes of bounded regions of a hyperplane arrangement. In Appendix B we pose some interesting questions. One concerning the relationship between singular toric metrics and toric *b*-divisors and another one concerning the Proj of the graded algebra coming from the global sections of multiples of a *b*-divisor. We also give some partial answers. In Appendix C we give a 3-dimensional example where we compute intersection numbers of differences of Cartier toric *b*-divisors.

Chapter 1

Intersection theory of *b*-divisors on toric varieties

The goal of this chapter is to develop an intersection theory of toric b-divisors on toric varieties. One of our main results concerns integrability of toric b-divisors. In particular, we show that a *nef* toric b-divisor is integrable and that its degree corresponds to the volume of a convex set. We also describe the class of convex sets which arise this way. We then proceed to describe the set of global sections of a (not necessarily nef) toric b-divisor in terms of lattice points in a convex set and, in the nef case, prove a Hilbert–Samuel type formula relating the degree with the asymptotic growth of the dimension of the space of global sections of multiples of the nef b-divisor. All of this generalizes classical results on toric varieties.

The concept of a toric b-divisor given in this thesis appears to be new as well as the theory of toric b-divisors developed in this chapter.

1.1 Notations and basic facts of toric geometry

Let k be an algebraically closed field of characteristic 0 and $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m}}^n$ an algebraic torus over k. An n-dimensional *toric variety* is a normal k-variety with an open subvariety isomorphic to the torus $\mathbb{T}$ and such that the action of the torus on itself can be extended to a regular action on X.

We start by recalling some notations which are standard in toric geometry and some basic facts of the classical theory. For a more detailed introductory text on toric geometry we refer to [CLS10] or [Ful93].

Polyhedral cones and fans

Let V be a real vector space. A *polyhedral cone* σ in V is the non-negative span of a finite set of vectors in V. A hyperplane H in V is a *supporting hyperplane* of σ if σ is contained in one of the two closed half spaces determined by H. The *span* of σ is the smallest subspace of V containing σ. The *relative interior of* σ, denoted by $\mathrm{relint}(\sigma)$ is the interior of σ in its span. A *face* τ of σ is the intersection of σ with any supporting hyperplane. We will use $\tau \leq \sigma$ to denote that τ is a face of σ. The cone σ is said to be *strongly convex* if it has $\{0\}$ as face. The *dual cone* $\sigma^\vee$ of σ is the subset of $V^\vee := \mathrm{Hom}(V, \mathbb{R})$ consisting of vectors $v^\vee$ which map σ to the non-negative reals. The dual cone $\sigma^\vee$ is also a polyhedral cone.

Let $N \simeq \mathbb{Z}^n$ be a lattice and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice. For every ring R, we denote by N_R or M_R the tensor product $N \otimes_{\mathbb{Z}} R$ or $M \otimes_{\mathbb{Z}} R$, respectively. A *strongly convex rational* *polyhedral cone* σ in $N_{\mathbb{R}}$ is a strongly convex polyhedral cone generated by vectors in N. In this case, the dual cone $\sigma^{\vee}$ is also a strongly convex rational polyhedral cone in $M_{\mathbb{R}}$.

A *rational polyhedral fan* Σ is a collection of strongly convex rational polyhedral cones closed under "$\leq$" and such that every two cones in Σ intersect along a common face. We always assume a fan Σ to be non-degenerate, i.e. it is not contained in any proper subspace of $N_{\mathbb{R}}$. From now on, we will just say *fan* and *cone* even though we mean "rational polyhedral fan" and "strongly convex rational polyhedral cone", respectively. We denote by $\Sigma(d)$ the set of d-dimensional cones in Σ.

Given a fan Σ in $N_{\mathbb{R}}$, a fan Σ' is a *subdivision* of Σ if every cone of Σ' is contained in a cone of Σ and $|\Sigma'| = |\Sigma|$. Hence, every cone of Σ is a union of cones of Σ'.

We denote by N^{prim} the set of primitive elements in N. Note that a primitive element v induces the ray, i.e. 1-dimensional cone $\mathbb{R}_{\geq 0}v$ in $N_{\mathbb{R}}$ which we denote by τ_v. Conversely, any ray τ in $N_{\mathbb{R}}$ is of the form $\tau_v = \mathbb{R}_{\geq 0}v_\tau$ for some primitive vector v_τ in N^{prim}.

Toric varieties and fans

Let N and M be as above. A fan Σ in $N_{\mathbb{R}}$ specifies a toric variety X_Σ as follows:

(1) For each cone σ in Σ, let $U_\sigma := \mathrm{Spec}\,(k\,[\sigma^{\vee} \cap M])$, where $k\,[\sigma^{\vee} \cap M]$ is the algebra attached to the semi-group $\sigma^{\vee} \cap M$.

(2) For a cone σ in Σ and each face $\tau \leq \sigma$ define $(U_\sigma)_\tau$ to be the spectrum of the localization $k\,[\sigma^{\vee} \cap M]_m$ where $m \in \sigma^{\vee} \cap M$ determines a supporting hyperplane for τ. Note that there is a canonical isomorphism $(U_\sigma)_\tau \simeq U_\tau$.

(3) For σ_1, σ_2 in Σ define $(U_{\sigma_1})_{\sigma_2} := (U_{\sigma_1})_{\sigma_1 \cap \sigma_2}$ and let X_Σ be the scheme obtained by gluing the schemes U_σ along the isomorphisms obtained by composing

$$(U_{\sigma_1})_{\sigma_2} \longrightarrow U_{\sigma_1 \cap \sigma_2} \longrightarrow (U_{\sigma_2})_{\sigma_1}.$$

It follows from the construction that X_Σ is separated, integral and normal. Moreover, the set $U_0 = \mathrm{Spec}\,(k\,[M]) \simeq (k^{\times})^n$ is an open subvariety of X_Σ henceforth denoted by $\mathbb{T}_N$. The action of the torus $\mathbb{T}_N$ on itself extends to a regular action on the whole of X_Σ which is given locally by the map $f\colon k\,[\sigma^{\vee} \cap M] \to k\,[\sigma^{\vee} \cap M] \otimes_k k[M]$ given by $m \mapsto m \otimes m$. Conversely, every toric variety can be specified by a fan as above.

The lattices M and N have natural geometric interpretations, namely M is the character lattice of the torus and its dual lattice N is the lattice of one-parameter subgroups of the torus. We will denote characters by χ^m for elements m in M. When the lattice N is clear from the context, we will just write $\mathbb{T}$ for $\mathbb{T}_N$. We will denote by $\langle \cdot, \cdot \rangle$ the pairing induced by the dual lattices M and N.

Many geometric properties of the toric variety X_Σ can be read off from combinatorial properties of the fans. For example, X_Σ is complete or smooth if and only if Σ is complete or smooth, respectively. A fan Σ is said to be *complete* if the set $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ is the whole of $N_{\mathbb{R}}$. The set $|\Sigma|$ is called the *support* of the fan Σ. The fan Σ is said to be *smooth* if each cone $\sigma \in \Sigma$ is *smooth*, i.e. if σ is spanned by part of a $\mathbb{Z}$-basis for N.

Toric morphisms

Let N_1 and N_2 be two lattices and let Σ_1 and Σ_2 be two fans in $N_{1\mathbb{R}}$ and $N_{2\mathbb{R}}$, respectively. Let π be an integral linear map from $N_{1\mathbb{R}}$ to $N_{2\mathbb{R}}$. We say that π is compatible with Σ_1 and Σ_2 if every cone $\sigma_1 \in \Sigma_1$ is mapped by π into a single cone $\sigma_2 \in \Sigma_2$, i.e. $\pi(\sigma_1) \subseteq \sigma_2$. A compatible map π

induces a morphism between the toric varieties X_{Σ_1} and X_{Σ_2} (which we also denote by π) satisfying the following two conditions: first we have $\pi(\mathbb{T}_{N_1}) \subseteq \mathbb{T}_{N_2}$ and the restriction $\pi|_{\mathbb{T}_{N_1}} : \mathbb{T}_{N_1} \to \mathbb{T}_{N_2}$ is a group homomorphism. Secondly, it is *equivariant*, meaning that $\pi(t \cdot p) = \phi(t) \cdot \pi(p)$ for all $t \in \mathbb{T}_{N_1}$ and $p \in X_{\Sigma_1}$. A morphism between two toric varieties satisfying the above properties is said to be *toric*. Conversely, one can show that any toric morphism between two toric varieties arises from an integral linear map between lattices.

A special class of toric morphisms is given by a subdivision $\Sigma' \subseteq N_{\mathbb{R}}$ of a given fan $\Sigma \subseteq N_{\mathbb{R}}$ with the identity as the lattice morphism. We will denote the resulting toric morphsim by $\pi_{\Sigma'} : X_{\Sigma'} \to X_{\Sigma}$. The toric morphism $\pi_{\Sigma'}$ arising in this way is proper and birational. Conversely, every proper, birational toric morphism to a toric variety X_{Σ} is given by a subdivision Σ' of Σ.

Stratification by toric orbits

Let $n = \dim(X_{\Sigma})$. The toric variety X_{Σ} has a natural stratification given by the cones σ in Σ. This stratification induces a bijective, order reversing correspondence between i-dimensional cones in Σ and $(n-i)$-dimensional toric subvarieties (i.e. subvarieties which are invariant under the action of the torus, or equivalently, which are toric varieties themselves).

This correspondence is as follows: for a cone $\sigma \in \Sigma$ we define the sets $N(\sigma) := N/(N \cap \mathbb{R}\sigma)$ and $M(\sigma) := N(\sigma)^{\vee} = M \cap \sigma^{\perp}$. Here, $\mathbb{R}\sigma$ is the linear space spanned by σ and $\sigma^{\perp}$ is the orthogonal complement of σ. We denote by $\pi_{\sigma} : N \to N(\sigma)$ the projection of lattices and also by $\pi_{\sigma} : N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$ the induced projection of vector spaces. Now, to each cone $\sigma \in \Sigma$, one can associate a distinguished closed point $x_{\sigma} \in X_{\Sigma}$. The torus orbit $O(\sigma)$ corresponding to the cone σ is the torus orbit of this distinguished point. We denote by $V(\sigma)$ the Zariski closure of $O(\sigma)$. The subvariety $V(\sigma)$ has a natural structure of a toric variety: consider the fan $\Sigma(\sigma)$ in $N(\sigma)_{\mathbb{R}}$ defined by

$$\Sigma(\sigma) := \{\pi_{\sigma}(\tau) \mid \tau \supseteq \sigma\}.$$

This fan is called the *star of* Σ *at* σ. Now, if we let $\tilde{\tau} := \pi_{\sigma}(\tau)$, we get closed immersions $X_{\tilde{\tau}} \hookrightarrow X_{\tau}$. These maps glue together to give a closed immersion $X_{\Sigma(\sigma)} \hookrightarrow X_{\Sigma}$ inducing an isomorphism $X_{\Sigma(\sigma)} \simeq V(\sigma)$. Note that by construction we have that $V(\sigma) = \overline{O(\sigma)} = \bigcup_{\tau \supseteq \sigma} O(\tau)$. Hence we indeed get a stratification of X_{Σ} by torus orbits.

Divisors on toric varieties

As a particular case of the above correspondence, we get a bijective correspondence between the set $\Sigma(1)$ of rays of the fan Σ and the set of 1-codimensional toric subvarieties of X_{Σ}. Moreover, the subvarieties $V(\tau)$ for $\tau \in \Sigma(1)$ generate the class group of the toric variety X_{Σ} denoted by $\mathrm{Cl}(X_{\Sigma})$. This follows from the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_{\Sigma}) \longrightarrow 0,$$

where $\mathbb{Z}^{\Sigma(1)}$ is the free group generated by the symbols y_{τ} for $\tau \in \Sigma(1)$. The first morphism takes $m \in M$ to its divisor $\mathrm{div}(\chi^m) = \sum_{\tau_v \in \Sigma(1)} \langle m, v \rangle y_{\tau_v}$. Here, v stands for the primitive vector spanning the ray τ_v. We will mostly be dealing with smooth varieties so that we have an isomorphism $\mathrm{Cl}(X_{\Sigma}) \simeq \mathrm{Pic}(X_{\Sigma})$.

We now give a combinatorial characterization of toric Cartier divisors. A *virtual support function* is a continuous function $\psi : |\Sigma| \to \mathbb{R}$ such that, for every cone $\sigma \in \Sigma$, there exists $m_{\sigma} \in M_{\mathbb{Q}}$ with $\psi(u) = \langle m_{\sigma}, u \rangle$ for all $u \in \sigma$. Now, let $\{m_{\sigma}\}_{\sigma \in \Sigma}$ be a set of defining vectors of a virtual support function ψ. On each open set U_{σ}, the vector m_{σ} determines a rational function $\chi^{-m_{\sigma}}$. For two cones σ and σ' in Σ, the rational function $\chi^{-m_\sigma}/\chi^{-m_{\sigma'}}$ is regular on $U_\sigma \cap U_{\sigma'} = U_{\sigma\cap\sigma'}$ and so ψ determines the Cartier divisor

$$D_\psi := \left\{ \left(U_\sigma, \chi^{-m_\sigma} \right) \right\}_{\sigma\in\Sigma}$$

on X_Σ. This Cartier divisor is toric and does not depend on the choice of defining vectors. Moreover, all toric Cartier divisors are obtained in this way. Note that a virtual support function on a complete fan is the support function of the polytope given as the convex hull $\operatorname{conv}\left(\{m_\sigma\}_{\sigma\in\Sigma(n)} \right) \subseteq M_\mathbb{R}$.

Assume that we start with a complete fan Σ. It turns out that convexity properties of virtual support functions encode positivity properties of the corresponding toric Cartier divisors: a function $f\colon N_\mathbb{R} \to \underline{\mathbb{R}}$ $(:= \mathbb{R} \cup \{-\infty\})$ is *concave* or *strictly concave*, if for all convex combinations $\sum_{i=1}^r a_i\, v_i \in N_\mathbb{R}$, i.e. the a_i's are non-negative real numbers satisfying $\sum_{i=1}^r a_i = 1$, the inequality

$$f\left(\sum_{i=1}^r a_i v_i\right) \geq \sum_{i=1}^r a_i f(v_i) \quad \text{or} \quad f\left(\sum_{i=1}^r a_i v_i\right) > \sum_{i=1}^r a_i f(v_i)$$

is respectively satisfied. The correspondence between Cartier toric divisors and virtual support functions induces a bijective correspondence between concave or strictly concave virtual support functions and nef or ample toric divisors, respectively.

The divisor of a polytope

Let $P \subseteq M_\mathbb{R}$ be a full dimensional *rational* polytope, i.e. the convex hull of finitely many elements in $M_\mathbb{Q}$. Let $F_1,\ldots,F_r$ be the facets of P with corresponding primitive, inward pointing normal vectors $v_1,\ldots,v_r$, respectively. Then there exist rational numbers $a_1,\ldots,a_r$ such that

$$P = \left\{ m \in M_\mathbb{R} \,\middle|\, \langle m, v_i\rangle \geq -a_i,\, \forall i \in \{1,\ldots,r\} \right\},$$

is the facet representation of P.

There is a dimension- and inclusion-reversing correspondence between faces of P and cones in $N_\mathbb{R}$ which sends a face Q of P to the polyhedral cone

$$\sigma_Q := \operatorname{cone}\left(v_i \,|\, F_i \text{ contains } Q \right).$$

The *normal fan* of Σ_P of P is defined by

$$\Sigma_P := \{ \sigma_Q \,|\, Q \text{ is a face of } P \}.$$

We will write $Q \leq P$ to denote that Q is a face of P. Note that facets of P correspond to rays in the normal fan Σ_P, so that each facet F gives a prime torus-invariant divisor $D_F \subseteq X_{\Sigma_P}$. Thus the facet representation of P gives the $\mathbb{Q}$-toric Weil divisor

$$D_P = \sum_F a_F D_F,$$

where the sum is over all facets of P. When it is clear from the context, we will refer to a rational polytope just as a polytope.

Global sections of toric divisors

To any toric Weil divisor $D = \sum_{\tau \in \Sigma(1)} a_\tau D_\tau$ on a complete toric variety X_Σ, one can associate the rational polytope

$$P_D := \{m \in M_\mathbb{R} \mid \langle m, v_\tau \rangle \geq -a_\tau\},$$

where v_τ is the primitive vector spanning the ray τ. We remark that the polytope of a divisor corresponding to a polytope is the polytope itself, i.e. we have that $P_{D_P} = P$.

If D is Cartier, then P_D encodes the global sections of the associated line bundle $\mathcal{O}(D)$. We have the following isomorphism:

$$\Gamma(X_\Sigma, \mathcal{O}(D)) \simeq \bigoplus_{m \in P_D \cap M} k \cdot \chi^{-m}.$$

The degree of toric nef divisors

If D is nef, then $P_D = \operatorname{conv}\left(\{m_\sigma\}_{\sigma \in \Sigma(n)}\right) \subseteq M_\mathbb{R}$, where the m_σ are the vectors determining the Cartier data of D. In this case, the degree of D is given by

$$D^n = n! \operatorname{vol}(P_D),$$

where the operator "vol" stands for the volume computed with respect to the Haar measure on $M_\mathbb{R}$ normalized so that the lattice M has covolume 1.

In terms of virtual support functions we have the following: let $f\colon N_\mathbb{R} \to \underline{\mathbb{R}}$ be a concave function. The *Legendre–Fenchel dual* of f is the function $f^\vee : M_\mathbb{R} \to \underline{\mathbb{R}}$ defined by the assignment

$$m \longmapsto \inf_{v \in N_\mathbb{R}} (\langle m, v \rangle - f(v)).$$

The *stability set* of f is defined to be the convex set Δ_f given by

$$\Delta_f := \operatorname{dom}(f^\vee) = \{m \in M_\mathbb{R} \mid \langle m, v \rangle - f(v) \text{ is bounded below for all } v \in N_\mathbb{R}\}.$$

In particular, if $f = \psi_D$ is the concave virtual support function associated to a nef toric divisor D, then we have that $\Delta_\psi = P_D$ and hence

$$D^n = n! \operatorname{vol}(P_D) = n! \operatorname{vol}(\Delta_\psi).$$

Mixed volumes and mixed degrees

One can generalize the above fact about the top self intersection number of a nef toric divisor to the mixed top intersection number of a collection of toric nef divisors $D_1, \ldots, D_n$. Indeed, the *mixed volume* $\operatorname{MV}(K_1, \ldots, K_n)$ of a collection of convex sets $K_1, \ldots, K_n$ is defined by

$$\operatorname{MV}(K_1, \ldots, K_n) := \sum_{j=1}^n (-1)^{n-j} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \operatorname{vol}\left(K_{i_1} + \cdots + K_{i_j}\right),$$

where the "+" refers to *Minkowski addition* of convex sets. Then, the mixed degree of $D_1, \ldots, D_n$ is equal to the mixed volume of the corresponding polytopes, i.e. we have

$$D_1 \cdots D_n = \operatorname{MV}(P_{D_1}, \ldots, P_{D_n}).$$

1.2 b-divisors on toric varieties

Throughout this section, $\Sigma \subseteq N_{\mathbb{R}}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. We define toric b-divisors on the toric variety X_Σ as a tower of toric Cartier $\mathbb{Q}$-divisors indexed over all toric birational morphisms and satisfying some compatibility condition. Moreover, as it is usual in toric geometry where geometric objects correspond to combinatorial ones, we will show that toric b-divisors correspond to conical, $\mathbb{Q}$-valued functions on the space $N_{\mathbb{Q}}$.

Let $\mathbb{T}\text{-Ca}\,(X_\Sigma)$ and $\mathbb{T}\text{-We}\,(X_\Sigma)$ be the space of toric Cartier and of toric Weil divisors, respectively. The word "toric" means that they are invariant under the action of the torus. We consider Cartier and Weil divisors with $\mathbb{Q}$-coefficients, i.e. the spaces $\mathbb{T}\text{-Ca}\,(X_\Sigma)_{\mathbb{Q}}$ and $\mathbb{T}\text{-We}\,(X_\Sigma)_{\mathbb{Q}}$, respectively.

Definition 1.2.1. The set $R\,(\Sigma)$ is defined to be the collection of smooth fans subdividing Σ. This forms a directed set under the following partial order

$$\Sigma'' \geq \Sigma' \quad \text{iff} \quad \Sigma'' \text{ is a smooth subdivision of } \Sigma'.$$

The *toric Riemann–Zariski space* of X_Σ is defined as the inverse limit

$$\mathfrak{X}_\Sigma := \varprojlim_{\Sigma' \in R(\Sigma)} X_{\Sigma'}$$

with maps given by the toric proper birational morphisms $X_{\Sigma''} \to X_{\Sigma'}$ induced whenever $\Sigma'' \geq \Sigma'$.

Remark 1.2.2. It is a standard fact in toric geometry that we have a bijective correspondence between fans $\Sigma' \in R\,(\Sigma)$ and pairs $(X_{\Sigma'}, \pi_{\Sigma'})$ of smooth and complete toric varieties $X_{\Sigma'}$ together with toric proper birational morphisms $\pi_{\Sigma'} \colon X_{\Sigma'} \to X_\Sigma$, up to isomorphism (see e.g. [CLS10]).

Definition 1.2.3. We define the group of *toric Cartier b-divisors* on X_Σ as the direct limit

$$\mathrm{Ca}(\mathfrak{X}_\Sigma)_{\mathbb{Q}} := \varinjlim_{\Sigma' \in R(\Sigma)} \mathbb{T}\text{-Ca}(X_{\Sigma'})_{\mathbb{Q}}$$

with maps given by the pull-back map of toric Cartier divisors. The group of *toric Weil b-divisors* is defined as the inverse limit

$$\mathrm{We}(\mathfrak{X}_\Sigma)_{\mathbb{Q}} := \varprojlim_{\Sigma' \in R(\Sigma)} \mathbb{T}\text{-Ca}(X_{\Sigma'})_{\mathbb{Q}}$$

with maps given by the push-forward map of toric Cartier divisors.

Remark 1.2.4. Our definition of toric Cartier and toric Weil b-divisors is inspired by Shokurov's b-divisors for which [BdFF12] constitutes a thorough reference. The difference is that in this thesis we consider only *toric* proper birational models which are somewhat governed by the combinatorics, whereas Shokurov's b-divisors are indexed over all proper birational models.

Notation. Since we will mostly deal with $\mathbb{Q}$-coefficients, we will omit the $\mathbb{Q}$ from our notations ($\mathbb{Q}$-coefficients always being implicit, unless stated otherwise). We will denote b-divisors in bold notation $\boldsymbol{D}$ to distinguish them from classical divisors D.

Remark 1.2.5. We make the following remarks.

(1) It follows from the definitions that a toric Weil b-divisor $\boldsymbol{D}$ on X_Σ consists of a net of toric Cartier $\mathbb{Q}$-divisors

$$\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}\,, \tag{1.1}$$

where $D_{\Sigma'} \in \mathbb{T}\text{-Ca}(X_{\Sigma'})$, which are compatible under push-forward, i.e. such that we have $\pi_* D_{\Sigma''} = D_{\Sigma'}$ whenever $\Sigma'' \geq \Sigma'$ and $\pi\colon X_{\Sigma''} \to X_{\Sigma'}$ is the corresponding toric morphism. For every $\Sigma' \in R(\Sigma)$, we say that $D_{\Sigma'}$ is the *incarnation* of $\boldsymbol{D}$ on the model $X_{\Sigma'}$. On the other hand, a toric Cartier b-divisor $\boldsymbol{E}$ on X_Σ is a toric Weil b-divisor

$$\boldsymbol{E} = (E_{\Sigma'})_{\Sigma' \in R(\Sigma)}\,, \tag{1.2}$$

for which there is a model $X_{\Sigma'}$ for a $\Sigma' \in R(\Sigma)$ such that for every other higher model $X_{\Sigma''}$, where $\Sigma'' \geq \Sigma'$ in $R(\Sigma)$, the incarnation $D_{\Sigma''}$ is the pullback of $D_{\Sigma'}$ on $X_{\Sigma'}$. In this case, we say that $\boldsymbol{E}$ is *determined* by Σ'. A toric Weil or Cartier b-divisor $\boldsymbol{D}$ or $\boldsymbol{E}$ is assumed to come always with a net as in (1.1) or (1.2), respectively.

(2) Since our fans are assumed to be smooth, we may identify Cartier and Weil toric divisors.

(3) We clearly have the containment $\mathrm{Ca}(\mathfrak{X}_\Sigma)_\mathbb{Q} \subseteq \mathrm{We}(\mathfrak{X}_\Sigma)_\mathbb{Q}$. Henceforth, we will refer to a *toric Weil b-divisor* simply as a *toric b-divisor*.

The next lemma gives the combinatorial analogues of toric b-divisors. Before stating it, we need the following two definitions.

Definition 1.2.6. Let $\Sigma' \in R(\Sigma)$ and let $\psi_{\Sigma'}$ be a virtual support function on the toric variety $X_{\Sigma'}$. Then for every fan $\tilde{\Sigma} \in R(\Sigma)$ such that $\tilde{\Sigma} \leq \Sigma'$ and inducing $\pi\colon X_{\Sigma'} \to X_{\tilde{\Sigma}}$, we will denote by $\pi_*(\psi_{\Sigma'})$ the piecewise linear function on $\tilde{\Sigma}$ defined by the values of $\psi_{\Sigma'}$ on the rays of $\tilde{\Sigma}$.

Definition 1.2.7. A function $f\colon N_\mathbb{R} \to \mathbb{R}$ is called *conical* if $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}_{\geq 0}$.

Lemma 1.2.8. *There is a natural bijective correspondence between the following objects:*

(1) Toric b-divisors on X_Σ.

(2) Collections $\{\psi_{\Sigma'}\}_{\Sigma' \in R(\Sigma)}$ of $\mathbb{R}$-valued functions on $N_\mathbb{R}$ satisfying the following two conditions:

(a) For each $\Sigma' \in R(\Sigma)$, the function $\psi_{\Sigma'}$ is a virtual support function for Σ'.

(b) For every toric birational morphism $\pi\colon X_{\Sigma''} \to X_{\Sigma'}$ induced by a regular subdivision $\Sigma'' \geq \Sigma'$ we have $\pi_*(\psi_{\Sigma''}) = \psi_{\Sigma'}$.

(3) Conical functions $\tilde{\phi}\colon N_\mathbb{Q} \to \mathbb{Q}$.

Proof. The bijective correspondence between (1) and (2) comes from the general theory of toric divisors and the fact that b-divisors are defined as a projective limit, hence requiring the compatibility condition in (2b).

To see that (2) implies (3) let $\{\psi_{\Sigma'}\}_{\Sigma' \in R(\Sigma)}$ be a collection as in (2). Note that a conical function on $N_\mathbb{Q}$ is determined by it values on the set of primitive elements which we denote by N^{prim}. Hence, let v be in N^{prim} and consider τ_v the ray spanned by v. Let $\Sigma' \in R(\Sigma)$ be any fan containing τ_v. We define $\tilde{\phi}(v) := \psi_{\Sigma'}(v)$. To see that this is well defined, let $\Sigma'' \in R(\Sigma)$ be any other fan containing τ_v. Let $\Sigma''' \in R(\Sigma)$ be a common refinement of Σ' and Σ''. Then, clearly, $\tau_v \in \Sigma'''$ and, denoting by $\pi'\colon X_{\Sigma'''} \to X_{\Sigma'}$ and by $\pi''\colon X_{\Sigma'''} \to X_{\Sigma''}$ the corresponding morphisms, we have

$$\psi_{\Sigma'}(v) = \pi'_*\psi_{\Sigma'''}(v) = \pi''_*\psi_{\Sigma'''}(v) = \psi_{\Sigma''}(v).$$

Now, to see that (3) implies (1) suppose that we are given a function $\tilde{\phi}\colon N^{\mathrm{prim}} \to \mathbb{Q}$. This induces for every $\Sigma' \in R(\Sigma)$ a toric Weil divisor

$$D_{\tilde{\phi},\Sigma'} := \sum_{\tau\in\Sigma'(1)} a_\tau D_\tau$$

on $X_{\Sigma'}$ by setting $a_\tau = -\tilde{\phi}(v_\tau)$ and hence, by smoothness, a toric Cartier divisor on each $X_{\Sigma'}$. The fact that this collection forms a toric b-divisor follows from the fact that for every toric morphsim $\pi\colon X_{\Sigma''} \to X_{\Sigma'}$ coming from a regular subdivision $\Sigma'' \geq \Sigma'$, we have

$$\pi_*\left(D_{\tilde{\phi},\Sigma''}\right) = \pi_*\left(\sum_{\tau\in\Sigma''(1)} a_\tau D_\tau\right) = \sum_{\tau\in\Sigma'(1)} a_\tau D_\tau = D_{\tilde{\phi},\Sigma'}. \qquad \blacksquare$$

Before we end this section, we define a special class of smooth subdivisions of the fan Σ. The following definitions are taken taken from [CLS10, Section 3.1.].

Definition 1.2.9. Let σ be a d-dimensional smooth cone in Σ. Let $\{v_1,\dots,v_d\}$ be generators of σ forming part of a $\mathbb{Z}$-basis of N. The *barycenter* v_σ of σ is defined to be the element

$$v_\sigma := \sum_{i=1}^{d} v_i.$$

Definition 1.2.10. Given a fan Σ and a smooth cone $\sigma \in \Sigma$, the *barycentric subdivision of* Σ *corresponding to the primitive element* v_σ, denoted by $\Sigma^*(v_\sigma)$, is the fan consisting of the following cones:

- $\tau \in \Sigma$, where $v_\sigma \notin \tau$.
- $\mathrm{cone}(\tau, v_\sigma)$ where $v_\sigma \notin \tau \in \Sigma$ and $\{v_\sigma \cup \tau\} \subseteq \sigma' \in \Sigma$.

If Σ is smooth, then $\Sigma^*(v_\sigma)$ is a smooth refinement of Σ. We also refer to $\Sigma^*(v_\sigma)$ as the *barycentric subdivision of* Σ *at the cone* σ.

Example 1.2.11. Consider the (non-complete) fan $\Sigma \subseteq \mathbb{R}^3$ consisting of the cone $\sigma = \mathbb{R}^3_{\geq 0}$ together with all of its faces. Figure 1.1 shows a 2-dimensional picture of two star subdivisions: one at the barycenter of σ, and one at the barycenter of the two dimensional cone τ.

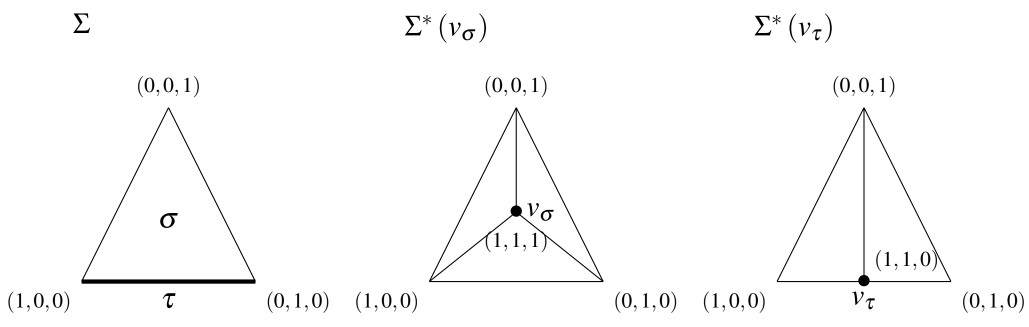


Figure 1.1: Star subdivisions of $\mathbb{R}^3_{\geq 0}$

We have the following two remarks.

Remark 1.2.12. Given any smooth subdivision Σ' of Σ we can always find a smooth fan $\tilde{\Sigma}$ obtained by a sequence of barycentric subdivisions of Σ which dominates Σ'. This follows from the fact that barycentric subdivisions correspond to blow ups of toric varieties along smooth torus invariant centers ([CLS10, Proposition 3.3.15]) and a particular case of the following De Concini–Procesi Theorem ([Bon00, Theorem 4.1]).

Theorem 1.2.13. *Let X and X' be smooth toric varieties with a toric birational map (not necessarily everywhere defined) $f\colon X \dashrightarrow X'$. Then, there exists a smooth toric variety $\tilde{X}$ obtained from X by a finite sequence $\pi\colon \tilde{X} \to X$ of blow-ups along smooth 2-codimensional invariant centers, such that $f \circ \pi$ is a regular map.*

Hence, the set of barycentric subdivisions of Σ along smooth cones is cofinal in $R(\Sigma)$.

Remark 1.2.14. We can give $R(\Sigma)$ the structure of a connected, oriented graph in the following way: let $\Sigma' \in R(\Sigma)$. If $\Sigma'' \in R(\Sigma)$ is such that $\Sigma'' = \Sigma'^{*}(v_\sigma)$ for a cone $\sigma \in \Sigma'$, then we put an arrow $\Sigma'' \to \Sigma'$.

To finalize this section, note that Lemma 1.2.8 implies that a priori we don't have any control over the coefficients of the prime toric divisors appearing in the different incarnations of a toric b-divisor as we move along the toric birational models. This is why, in general, in order to prove integrability results about toric b-divisors, we need to impose some positivity condition. It turns out, as it will become clear in the following section, that the right positivity condition is *nefness*.

1.3 Integrability of toric b-divisors

Throughout this section, $\Sigma \subseteq N_{\mathbb{R}}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. We define the mixed degree of a toric b-divisor. We further investigate the question regarding both necessary and sufficient conditions for integrability (i.e. existence and finiteness of the mixed degree). We give a partial answer to this question in the general case of dimension n and, in the following section, a complete answer in the case of toric surfaces. Specifically, in the n-dimensional case, we show that any collection of n nef toric b-divisors (and any difference of such) is integrable and give a formula for its mixed degree as the mixed volume of bounded convex sets (and differences of such). As a corollary we get a Brunn–Minkowski type inequality for the degree of nef toric b-divisors. We note that in Appendix A we describe what would be a starting point to compute top intersection numbers of (not necessarily nef nor difference of nef) toric b-divisors.

Before introducing the definition of the *mixed degree* of a collection of toric b-divisors, note that there is an obvious intersection pairing

$$\underbrace{\mathrm{Ca}(\mathfrak{X}_\Sigma) \times \cdots \times \mathrm{Ca}(\mathfrak{X}_\Sigma)}_{n\text{-times}} \longrightarrow \mathbb{Q}$$

of toric Cartier b-divisors defined as follows: if $(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_2) \in \mathrm{Ca}(\mathfrak{X}_\Sigma) \times \cdots \times \mathrm{Ca}(\mathfrak{X}_\Sigma)$, then let $\Sigma' \in R(\Sigma)$ be sufficiently large such that $\boldsymbol{E}_i$ is determined on Σ' for all $i = 1, \ldots, n$. We then set

$$\boldsymbol{E}_1 \cdots \boldsymbol{E}_n := E_{1,\Sigma'} \cdots E_{n,\Sigma'}.$$

This is independent of the choice of a common determination Σ' and hence the pairing given by

$$(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_n) \longmapsto \boldsymbol{E}_1 \cdots \boldsymbol{E}_n$$

is well defined. Moreover, we can extend the above intersection pairing to a pairing

$$\underbrace{\mathrm{Ca}(\mathfrak{X}_\Sigma)\times\cdots\times\mathrm{Ca}(\mathfrak{X}_\Sigma)\times\mathrm{We}(\mathfrak{X}_\Sigma)}_{n\text{-times}}\longrightarrow\mathbb{Q}$$

in the following way: if $(\boldsymbol{E}_1,\dots,\boldsymbol{E}_{n-1},\boldsymbol{D})\in\mathrm{Ca}(\mathfrak{X}_\Sigma)\times\cdots\times\mathrm{Ca}(\mathfrak{X}_\Sigma)\times\mathrm{We}(\mathfrak{X}_\Sigma)$, we let Σ' be a common determination of $\boldsymbol{E}_i$ for $i=1,\dots,n-1$. We then set

$$\boldsymbol{E}_1\cdots\boldsymbol{E}_{n-1}\cdot\boldsymbol{D}:=E_{1,\Sigma'}\cdots E_{n-1,\Sigma'}\cdot D_{\Sigma'}.$$

Using the projection formula, one can see that the above pairing is independent of the choice of a common determination Σ' and hence is well defined. Indeed, let $\Sigma''\in R(\Sigma)$ be another common determination of the $\boldsymbol{E}_i$'s. W.l.o.g. assume that $\Sigma''\geq\Sigma'$ and let $\pi_{\Sigma''}\colon X_{\Sigma''}\to X_{\Sigma'}$ the induced toric, proper, birational morphism. We get

$$\begin{aligned}E_{1,\Sigma''}\cdots E_{n-1,\Sigma''}\cdot D_{\Sigma''}&=\pi_{\Sigma''}^*\left(E_{1,\Sigma'}\right)\cdots\pi_{\Sigma''}^*\left(E_{n-1,\Sigma'}\right)\cdot D_{\Sigma''}\\&=\pi_{\Sigma''}^*\left(E_{1,\Sigma'}\cdots E_{n-1,\Sigma'}\right)\cdot D_{\Sigma''}\\&=E_{1,\Sigma'}\cdots E_{n-1,\Sigma'}\cdot\pi_{\Sigma''*}\left(D_{\Sigma''}\right)\\&=E_{1,\Sigma'}\cdots E_{n-1,\Sigma'}\cdot D_{\Sigma'},\end{aligned}$$

where we have used the projection formula to pass from the second to the third line. However, in general, there is no obvious way to define an intersection pairing for more than one element in $\mathrm{We}(\mathfrak{X}_\Sigma)$. The following definition is inspired by the idea of "integrability" of a function (see [BKK16, Definition 3.17]). Recall that $R(\Sigma)$ is a directed set. In particular, it is a *net* and we have a well defined notion of limit.

Definition 1.3.1. Let $\boldsymbol{D}_1,\dots,\boldsymbol{D}_n$ be a collection of toric b-divisors. We define its *mixed degree* to be the limit

$$\boldsymbol{D}_1\cdots\boldsymbol{D}_n:=\lim_{\Sigma'\in R(\Sigma)}D_{1,\Sigma'}\cdots D_{n,\Sigma'},$$

if it exists and is finite. If $\boldsymbol{D}=\boldsymbol{D}_1=\cdots=\boldsymbol{D}_n$, then we call

$$\boldsymbol{D}^n=\lim_{\Sigma'\in R(\Sigma)}D^n_{\Sigma'},$$

if the limit exists and is finite, the *degree* of the b-divisor $\boldsymbol{D}$. A toric b-divisor whose degree exists and is finite is called *integrable*.

One of the main results of this section is that the mixed degree of a collection of *nef* toric b-divisors (see Definition 1.3.2) exists. In particular, a nef toric b-divisor is automatically integrable.

Integrability of nef toric b-divisors

Definition 1.3.2. A toric b-divisor $\boldsymbol{D}=(D_{\Sigma'})_{\Sigma'\in R(\Sigma)}$ is called *nef* if there exists a cofinal subset $S\subseteq R(\Sigma)$ such that $D_{\Sigma'}$ is nef in $X_{\Sigma'}$ for all $\Sigma'\in S$.

Remark 1.3.3. With notations as above, let $\psi_{\Sigma'}=\psi_{D_{\Sigma'}}\colon N_{\mathbb{R}}\to\mathbb{R}$ be the virtual support function corresponding to $D_{\Sigma'}$. Recall that, if $D_{\Sigma'}=\sum_{\tau\in\Sigma'(1)}a_\tau D_\tau$, then $\psi_{\Sigma'}$ is characterized by satisfying $\psi_{\Sigma'}(v_\tau)=-a_\tau$, where v_τ denotes the primitive vector spanning the ray τ. Hence, if $\Sigma''\geq\Sigma'$ and the map $\pi\colon X_{\Sigma''}\to X_{\Sigma'}$ is the associated toric morphism, then

$$D_{\Sigma''}\leq\pi^*D_{\Sigma'}\quad\text{ iff }\quad\psi_{\Sigma''}(v_\tau)\geq\psi_{\Sigma'}(v_\tau)$$

for every $\tau\in\Sigma''(1)$.

Now recall that classical nef toric divisors are in bijective correspondence with virtual support functions which have the additional property of being *concave*. Hence, using Remark 1.3.3, one expects that monotonicity properties of nef toric divisors follow directly from such analogous properties of concave functions. Indeed, we have the following lemma.

Lemma 1.3.4. *Let $\boldsymbol{D}$ be a nef toric b-divisor on X_Σ and let Σ' be a fan in $R(\Sigma)$. Then, for a sufficiently large $\Sigma'' \geq \Sigma'$ in $R(\Sigma)$, we have that $D_{\Sigma''} \leq \pi_{\Sigma''}^* D_{\Sigma'}$. Here, $\pi_{\Sigma''} \colon X_{\Sigma''} \to X_{\Sigma'}$ denotes the induced toric, proper, birational morphism.*

Proof. Let $\Sigma'' \geq \Sigma'$ be in $R(\Sigma)$ such that $D_{\Sigma''}$ is nef and hence $\psi_{\Sigma''}$ is concave. By Remark 1.3.3, and using the same notation therein, it suffices to show that for all $\tau \in \Sigma''(1)$ the inequality

$$\psi_{\Sigma''}(v_\tau) \geq \psi_{\Sigma'}(v_\tau) \tag{1.3}$$

is satisfied. Now, if $\tau \in \Sigma'(1)$ then $\psi_{\Sigma''}(v_\tau) = \psi_{\Sigma'}(v_\tau)$. Otherwise, assume that $\tau \in \Sigma''(1) \setminus \Sigma'(1)$. Let $\tau_1, \ldots, \tau_r \in \Sigma'(1)$ such that $\sigma := \operatorname{cone}\langle \tau_1, \ldots, \tau_r \rangle \in \Sigma'$ and $\tau \in \operatorname{relint}(\sigma)$. Then we can write

$$v_\tau = \sum_{i=1}^r a_i v_{\tau_i} \quad \text{with} \quad a_i > 0.$$

Hence, we have

$$\psi_{\Sigma''}(v_\tau) = \psi_{\Sigma''}\left(\sum_{i=1}^r a_i v_{\tau_i}\right) \geq \sum_{i=1}^r a_i \psi_{\Sigma''}(v_{\tau_i}) = \sum_{i=1}^r a_i \psi_{\Sigma'}(v_{\tau_i}) = \psi_{\Sigma'}(v_\tau).$$

Note that the second inequality follows by the concavity of $\psi_{\Sigma''}$. Thus, the inequality (1.3) is satisfied and thus the statement of the lemma follows. ∎

Next, we will show that this monotonicity property extends to intersection products. Recall the definition of the rational polytope P_D associated to a toric divisor D from Section 1.1.

Remark 1.3.5. Lemma 1.3.4 and its proof imply that if $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ is a nef toric b-divisor, then for every fan Σ' in $R(\Sigma)$, there exists a sufficiently large refinement $\Sigma'' \geq \Sigma'$ in $R(\Sigma)$ such that

$$\Sigma'' \geq \Sigma' \quad \text{iff} \quad P_{D''} \subseteq P_{D'}.$$

We can now state the aimed monotonicity property for intersection products of nef toric b-divisors.

Lemma 1.3.6. *Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ be nef toric b-divisors. Then for every fan Σ' in $R(\Sigma)$, there exists a sufficiently large refinement $\Sigma'' \geq \Sigma'$ in $R(\Sigma)$ such that the inequality*

$$D_{1,\Sigma''} \cdots D_{n,\Sigma''} \leq D_{1,\Sigma'} \cdots D_{n,\Sigma'}$$

holds true.

Proof. Recall from Section 1.1, that the mixed degree of a collection of toric nef divisors $D_1, \ldots, D_n$ is given by the mixed volume of their corresponding polytopes, i.e. we have

$$D_1 \cdots D_n = \operatorname{MV}(P_{D_1}, \ldots, P_{D_n}).$$

Hence, by Remark 1.3.5, for a large enough refinement $\Sigma'' \geq \Sigma'$ in $R(\Sigma)$, we have that

$$D_{1,\Sigma''} \cdots D_{n,\Sigma''} = \operatorname{MV}\left(P_{D_{1,\Sigma''}}, \ldots, P_{D_{n,\Sigma''}}\right) \leq \operatorname{MV}\left(P_{D_{1,\Sigma'}}, \ldots, P_{D_{n,\Sigma'}}\right) = D_{1,\Sigma'} \cdots D_{n,\Sigma'}. \qquad ∎$$

Remark 1.3.7. It follows from Lemma 1.2.8 that a nef toric b-divisor corresponds to a conical $\mathbb{Q}$-concave function $\tilde{\phi}\colon N_{\mathbb{Q}} \to \mathbb{Q}$. Here, by $\mathbb{Q}$-concavity we mean that for all convex combinations $\sum_{i=1}^{r} a_i v_i \in N_{\mathbb{Q}}$, i.e. the a_i's are in $\mathbb{Q}_{\geq 0}$ and satisfy $\sum_{i=1}^{r} a_i = 1$, the inequality

$$\tilde{\phi}\left(\sum_{i=1}^{r} a_i v_i\right) \geq \sum_{i=1}^{r} a_i \tilde{\phi}\left(v_i\right)$$

holds.

Before giving the main result of this section, we have a technical lemma.

Lemma 1.3.8. *Let $\tilde{\phi}\colon N_{\mathbb{Q}} \to \mathbb{Q}$ be a $\mathbb{Q}$-concave function and let $\bar{x} \in N_{\mathbb{R}}$. Then we have*

(1) There exist two positive real constants C and r such that $|\tilde{\phi}(x)| \leq C$ for all $x \in B(\bar{x}, r)$, where $B(\bar{x}, r) := \{x \in N_{\mathbb{Q}} \,\big|\, |x - \bar{x}| \leq r\}$.

(2) The function $\tilde{\phi}$ is Lipschitz continuous at $\bar{x}$, i.e. there exist two positive real constants L and r such that

$$|\tilde{\phi}(x) - \tilde{\phi}(x')| \leq L|x - x'|$$

for all $x, x' \in B(\bar{x}, r)$.

Proof. In order to prove part 1 of the lemma, we consider an n-dimensional simplex Δ with vertices $x_0, \ldots, x_n \in N_{\mathbb{Q}}$ containing $\bar{x}$ in its interior, i.e.

$$\bar{x} = \sum_{i=0}^{n} a_i x_i$$

with $a_i \in \mathbb{R}_{>0}$ such that $\sum_i a_i = 1$. Since Δ is top-dimensional, it contains a ball $B(\bar{x}, r)$ for some suitable small radius $r > 0$. We show that the function $\tilde{\phi}$ is bounded from below in $B(\bar{x}, r)$ by the quantity $\min_{0 \leq i \leq n} \tilde{\phi}(x_i)$. Indeed, let $x \in B(\bar{x}, r) \subseteq \Delta \cap N_{\mathbb{Q}}$. Then x is of the form $x = \sum_{i=0}^{n} b_i x_i$ with $b_i \in \mathbb{Q}_{\geq 0}$ satisfying $\sum_{i=0}^{n} b_i = 1$. By the $\mathbb{Q}$-concavity, we get

$$\tilde{\phi}(x) = \tilde{\phi}\left(\sum_{i=0}^{n} b_i x_i\right) \geq \sum_{i=0}^{n} b_i \tilde{\phi}(x_i) \geq \min_{0 \leq i \leq n} \tilde{\phi}(x_i).$$

Letting $C' := \min_{0 \leq i \leq n} \tilde{\phi}(x_i)$, we have shown that $\tilde{\phi}$ is bounded from below by C' in $B_{\bar{x}, r}$ as claimed. Now, we prove that $\tilde{\phi}$ is bounded from above in $B(\bar{x}, r)$ for the same radius r. For this, let $x \in B(\bar{x}, r)$ and set $x' = \bar{x} - (x - \bar{x}) = 2\bar{x} - x$. We get $|x' - \bar{x}| = |x - \bar{x}| \leq r$ and thus $x' \in B(\bar{x}, r)$. Since $\bar{x} = \frac{1}{2}(x + x')$, the $\mathbb{Q}$-concavity of $\tilde{\phi}$ shows that

$$2\tilde{\phi}(\bar{x}) \geq \tilde{\phi}(x) + \tilde{\phi}(x').$$

Therefore, since $\tilde{\phi}(x') \geq C'$, we get

$$\tilde{\phi}(x) \leq 2\tilde{\phi}(\bar{x}) - \tilde{\phi}(x') \leq 2\tilde{\phi}(\bar{x}) - C'.$$

By taking $C := \max\left\{C', 2\tilde{\phi}(\bar{x}) - C'\right\}$, the proof of part 1 is complete. Now, let us see that part 2 of the lemma is a direct consequence of part 1. Indeed, we show that $\tilde{\phi}$ is Lipschitz continuous in the neighborhood $B(\bar{x}, r/2)$. Let $x, x' \in B(\bar{x}, r/2)$ with $x \neq x'$. There exists an $x'' \in N_{\mathbb{Q}}$ such that the convex linear combination

$$x' = \lambda x'' + (1 - \lambda)x \ \text{ with } \ \lambda = \frac{|x' - x|}{|x'' - x|} \in (0,1) \cap \mathbb{Q} \quad \text{and} \quad \frac{2r}{3} \leq |x'' - \bar{x}| \leq r.$$

By $\mathbb{Q}$-concavity of $\tilde{\phi}$ we obtain

$$\tilde{\phi}(x)-\tilde{\phi}(x')=\tilde{\phi}(x)-\tilde{\phi}(\lambda x''+(1-\lambda)x)\leq\lambda\left(\tilde{\phi}(x)-\tilde{\phi}(x'')\right)=|x'-x|\frac{\tilde{\phi}(x)-\tilde{\phi}(x'')}{|x''-x|}.$$

Since $x,x''\in B(\bar{x},r)$, from part 1 we have that $|\tilde{\phi}(x)-\tilde{\phi}(x'')|\leq 2C$. Furthermore, we have

$$|x''-x|\geq|x''-\bar{x}|-|x-\bar{x}|\geq\frac{2r}{3}-\frac{r}{2}=\frac{r}{6},$$

which gives

$$|\tilde{\phi}(x)-\tilde{\phi}(x')|\leq\frac{12C}{r}|x-x'|,$$

as claimed in part 2 of the lemma. ■

Definition 1.3.9. A function $h\colon N_{\mathbb{R}}\to\mathbb{R}$ is called *rational* if it satisfies that $h\left(N_{\mathbb{Q}}\right)\subseteq\mathbb{Q}$.

The following is one of the main results of this section.

Theorem 1.3.10. *Let $\boldsymbol{D}$ be a nef toric b-divisor and let $\tilde{\phi}$ be its corresponding $\mathbb{Q}$-concave, conical function (see Remark 1.3.7). Then there existes a unique continuous, rational and concave function $\phi\colon N_{\mathbb{R}}\to\mathbb{R}$ extending $\tilde{\phi}$. Moreover, $\boldsymbol{D}$ is integrable and its degree equals $n!$ times the volume of the stability set of ϕ, i.e*

$$\boldsymbol{D}^n=n!\operatorname{vol}\left(\Delta_\phi\right).$$

Proof. Let $\boldsymbol{D}$ and $\tilde{\phi}$ be as in the statement of the theorem. Consider an element $v\in N_{\mathbb{R}}$ and let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in $N_{\mathbb{Q}}$ converging to v. We define the extension $\phi\colon N_{\mathbb{R}}\to\mathbb{R}$ by setting

$$\phi(v):=\lim_i\tilde{\phi}(v_i).$$

This is well defined by Lemma 1.3.8. Moreover, we claim that for every $v\in N_{\mathbb{R}}$ we have that

$$\lim_{\Sigma'\in R(\Sigma)}\psi_{\Sigma'}(v)=\phi(v).$$

In particular, this means that for all $v\in N_{\mathbb{R}}$, the sequence $\{\psi_{\Sigma'}(v)\}_{\Sigma'\in R(\Sigma)}$ is Cauchy. To prove the claim, note that if $v\in N_{\mathbb{Q}}$, then $\phi(v)=\tilde{\phi}(v)=\psi_{\tilde{\Sigma}}(v)$ for some $\tilde{\Sigma}\in R(\Sigma)$. Thus, we have that

$$\lim_{\Sigma'\in R(\Sigma)}\psi_{\Sigma'}(v)=\psi_{\tilde{\Sigma}}(v)=\tilde{\phi}(v)=\phi(v).$$

If $v\notin N_{\mathbb{Q}}$, by continuity, we get

$$\lim_{\Sigma'\in R(\Sigma)}\psi_{\Sigma'}(v)=\lim_i\lim_{\Sigma'\in R(\Sigma)}\psi_{\Sigma'}(v_i)=\lim_i\tilde{\phi}(v_i)=\phi(v),$$

thus proving the claim. Now, let $\Sigma''\geq\Sigma'\in R(\Sigma)$. Recalling the definition of the polytopes $P_{D_{\Sigma'}}$ and $P_{D_{\Sigma''}}$ (see Section 1.1), we have the following description of the complement $P_{D_{\Sigma'}}\setminus P_{D_{\Sigma''}}$:

$$\begin{aligned}P_{D_{\Sigma'}}\setminus P_{D_{\Sigma''}}&=\left\{m\in M_{\mathbb{R}}\,\middle|\,m\in P_{D_{\Sigma'}},\,m\notin P_{D_{\Sigma''}}\right\}\\&=P_{D_{\Sigma'}}\cap\left\{m\in M_{\mathbb{R}}\,\middle|\,\exists\tau''\in\Sigma''(1)\setminus\Sigma'(1)\colon\langle m,v_{\tau''}\rangle<\psi_{\Sigma''}(v_{\tau''})\right\}\\&=P_{D_{\Sigma'}}\cap\left\{m\in M_{\mathbb{R}}\,\middle|\,\exists\tau''\in\Sigma''(1)\setminus\Sigma'(1)\colon\psi_{\Sigma'}(v_{\tau''})\leq\langle m,v_{\tau''}\rangle<\psi_{\Sigma''}(v_{\tau''})\right\}.\end{aligned}$$

Hence, since the sequence $\{\psi_{\Sigma'}(v_{\tau''})\}_{\Sigma'\in R(\Sigma)}$ is Cauchy, we have that there exists a refinement $\tilde{\Sigma}\in R(\Sigma)$ such that for all $\Sigma',\Sigma''\geq\tilde{\Sigma}\in R(\Sigma)$ and for any $\varepsilon>0$, the inequality

$$\mathrm{vol}\left(P_{D_{\Sigma'}}\setminus P_{D_{\Sigma''}}\right)\leq\varepsilon$$

is satisfied. Therefore, by the monotonicity property in Lemma 1.3.6, we have that

$$\lim_{\Sigma'\in R(\Sigma)}\mathrm{vol}\left(P_{D_{\Sigma'}}\right)=\mathrm{vol}\left(\Delta_{\phi}\right).$$

Finally, the limit $\boldsymbol{D}^n$ exists and is given by

$$\boldsymbol{D}^n=\lim_{\Sigma'\in R(\Sigma)}D_{\Sigma'}^n=n!\lim_{\Sigma'\in R(\Sigma)}\mathrm{vol}\left(P_{D_{\Sigma'}}\right)=n!\,\mathrm{vol}\left(\Delta_{\phi}\right),$$

concluding the proof of the theorem. ∎

Corollary 1.3.11. *Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be nef toric b-divisors and suppose that the induced convex sets Δ_{D_1} and Δ_{D_2} from Theorem 1.3.10 are full-dimensional. Then, we have a Brunn–Minkowski-type inequality for their degree, i.e. the inequality*

$$(\boldsymbol{D}_1^n)^{1/n}+(\boldsymbol{D}_2^n)^{1/n}\leq((\boldsymbol{D}_1+\boldsymbol{D}_2)^n)^{1/n}$$

is satisfied.

Proof. This is a standard fact in convex geometry about volumes of convex sets (see e.g. [Gar02]). ∎

Our next goal is to show that the mixed degree of a collection of nef toric b-divisors $\boldsymbol{D}_1,\ldots,\boldsymbol{D}_n$ exists. We start with the following lemma:

Lemma 1.3.12. *For variables $X_1,\ldots,X_n$ the following holds:*

$$n!\,X_1\cdots X_n=\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{n-\#I}\left(\sum_{i\in I}X_i\right)^n.$$

Proof. This follows by induction on n. ∎

Theorem 1.3.13. *Let $\boldsymbol{D}_1,\ldots,\boldsymbol{D}_n$ be a collection of nef toric b-divisors. Then, the mixed degree*

$$\lim_{\Sigma'\in R(\Sigma)}D_{1,\Sigma'}\cdots D_{n,\Sigma'}$$

exists, is finite, and is given by the mixed volume of the stability sets of the corresponding concave functions $\phi_1,\ldots,\phi_d$, i.e. the equality

$$D_1\cdots D_n=MV\left(\Delta_{\phi_1},\ldots,\Delta_{\phi_n}\right)$$

holds true. Thus we have a well defined, multilinear map from the set of n-tuples of nef toric b-divisors to the non-negative real numbers which sends such an n-tuple to its mixed degree.

Proof. We show that the sequence $\left\{D_{1,\Sigma'}\cdots D_{n,\Sigma'}\right\}_{\Sigma'\in R(\Sigma)}$ is Cauchy. Let $\varepsilon>0$. For all subsets $I\subseteq\{1,\dots,n\}$, we let $\boldsymbol{D}_I:=\sum_{i\in I}\boldsymbol{D}_i$. Note that $\boldsymbol{D}_I$ is a positive linear combination of nef toric b-divisors and is thus nef. It follows from Theorem 1.3.10 that $\boldsymbol{D}_I$ is integrable. In particular, this means that the sequence $\left\{D^n_{I,\Sigma'}\right\}_{\Sigma'\in R(\Sigma)}$ is Cauchy. Hence, there exists a fan $\tilde{\Sigma}_I\in R(\Sigma)$ such that for all refinements $\Sigma',\Sigma''\geq\tilde{\Sigma}_I\in R(\Sigma)$, the inequality

$$\left|D^n_{I,\Sigma''}-D^n_{I,\Sigma'}\right|\leq\frac{n!\varepsilon}{2^n}$$

is satisfied. Let $\tilde{\Sigma}:=\max_{I\subseteq\{1,\dots,n\}}\tilde{\Sigma}_I$. Then, using Lemma 1.3.12, we get that for all $\Sigma',\Sigma''\geq\tilde{\Sigma}$, the sequence of inequalities

$$\begin{aligned}
&\left|D_{1,\Sigma''}\cdots D_{n,\Sigma''}-D_{1,\Sigma'}\cdots D_{n,\Sigma'}\right|\\
&=\left|\frac{1}{n!}\sum_{I\subseteq\{1,\dots,n\}}(-1)^{n-\#I}\left(\sum_{i\in I}D_{i,\Sigma''}\right)^n-\frac{1}{n!}\sum_{I\subseteq\{1,\dots,n\}}(-1)^{n-\#I}\left(\sum_{i\in I}D_{i,\Sigma'}\right)^n\right|\\
&=\frac{1}{n!}\left|\sum_{I\subseteq\{1,\dots,n\}}(-1)^{n-\#I}\left[\left(\sum_{i\in I}D_{i,\Sigma''}\right)^n-\left(\sum_{i\in I}D_{i,\Sigma'}\right)^n\right]\right|\\
&=\frac{1}{n!}\left|\sum_{I\subseteq\{1,\dots,n\}}(-1)^{n-\#I}\left[D^n_{I,\Sigma''}-D^n_{I,\Sigma'}\right]\right|\\
&\leq\frac{1}{n!}\sum_{I\subseteq\{1,\dots,n\}}\left|D^n_{I,\Sigma''}-D^n_{I,\Sigma'}\right|\leq\varepsilon
\end{aligned}$$

holds true, as we wanted to show. Moreover, we have that

$$\begin{aligned}
\boldsymbol{D}_1\cdots\boldsymbol{D}_n&=\lim_{\Sigma'\in R(\Sigma)}D_{1,\Sigma'}\cdots D_{n,\Sigma'}\\
&=\lim_{\Sigma'\in R(\Sigma)}\mathrm{MV}\left(P_{D_{1,\Sigma'}},\dots,P_{D_{n,\Sigma'}}\right)\\
&=\mathrm{MV}\left(\Delta_{\phi_1},\dots,\Delta_{\phi_n}\right),
\end{aligned}$$

concluding the proof of the theorem. ∎

As a corollary we get that also the difference of nef toric b-divisors is integrable.

Theorem 1.3.14. *Let $\boldsymbol{D}=\boldsymbol{D}_1-\boldsymbol{D}_2$ be the difference of two nef toric b-divisors $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. Then $\boldsymbol{D}$ is integrable and its degree is given as a sum of mixed volumes of the stability sets of the concave functions ϕ_1 and ϕ_2 corresponding to $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively.*

Proof. We show that the sequence $\left\{D^n_{\Sigma'}\right\}_{\Sigma'\in R(\Sigma)}=\left\{\left(D_{1,\Sigma'}-D_{2,\Sigma'}\right)^n\right\}_{\Sigma'\in R(\Sigma)}$ is Cauchy. Let $\varepsilon>0$. By the proof of Theorem 1.3.13, the nefness of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ implies that for all $i=1,\dots,n$ there exists a fan $\tilde{\Sigma}_i\in R(\Sigma)$ such that for all refinements $\Sigma',\Sigma''\geq\tilde{\Sigma}_i$, the inequality

$$\left|D^{n-i}_{1,\Sigma''}D^i_{2,\Sigma''}-D^{n-i}_{1,\Sigma'}D^i_{2,\Sigma'}\right|\leq\frac{\varepsilon}{(n+1)\binom{n}{i}}$$

holds true. Letting $\tilde{\Sigma} := \max_{i\in\{1,\dots,n\}} \tilde{\Sigma}_i$, we get

$$\begin{aligned}
|D^n_{\Sigma''} - D^n_{\Sigma'}| &= \left|(D_{1,\Sigma''} - D_{2,\Sigma''})^n - (D_{1,\Sigma'} - D_{2,\Sigma'})^n\right| \\
&= \left|\sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(D^{n-i}_{1,\Sigma''} D^i_{2,\Sigma''} - D^{n-i}_{1,\Sigma'} D^i_{2,\Sigma'}\right)\right| \\
&\leq \sum_{i=0}^{n} \binom{n}{i} \left|D^{n-i}_{1,\Sigma''} D^i_{2,\Sigma''} - D^{n-i}_{1,\Sigma'} D^i_{2,\Sigma'}\right| \leq \varepsilon
\end{aligned}$$

for all $\Sigma', \Sigma'' \geq \tilde{\Sigma}$, as claimed. Moreover, we get

$$\begin{aligned}
\boldsymbol{D}^n &= \lim_{\Sigma' \in R(\Sigma)} D^n_{\Sigma'} \\
&= \lim_{\Sigma' \in R(\Sigma)} \left(D_{1,\Sigma'} - D_{2,\Sigma'}\right)^n \\
&= \lim_{\Sigma' \in R(\Sigma)} \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{n-i}_{1,\Sigma'} D^i_{2,\Sigma'} \\
&= \lim_{\Sigma' \in R(\Sigma)} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathrm{MV}\left(\underbrace{P_{D_{1,\Sigma'}}, \dots, P_{D_{1,\Sigma'}}}_{(n-i)\text{-times}}, \underbrace{P_{D_{2,\Sigma'}}, \dots, P_{D_{2,\Sigma'}}}_{i\text{-times}}\right) \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathrm{MV}\left(\underbrace{\Delta_{\phi_1}, \dots, \Delta_{\phi_1}}_{(n-i)\text{-times}}, \underbrace{\Delta_{\phi_2}, \dots, \Delta_{\phi_2}}_{i\text{-times}}\right)
\end{aligned}$$

as stated in the theorem. ■

1.4 The case $n = 2$

In this section, we consider the case of smooth and complete toric surfaces, i.e. $\Sigma \subseteq N_{\mathbb{R}}$ is a smooth and complete fan of dimension 2. We see that integrability of a toric b-divisor on a toric surface is equivalent to the convergence of a certain series, where the sum ranges over pairs of relatively prime integers. We also provide some examples.

We start by fixing an identification of lattices $N \simeq \mathbb{Z}^2$ and we consider a primitive vector $v = (v_1, v_2) \in N^{\mathrm{prim}}$. We denote by P_v be the Newton polyhedron associated to the monomial ideal $I = (x^{v_2}, y^{v_1})$. Note that P_v is not bounded. Also, we let $\Sigma_{P_v} \subseteq N_{\mathbb{R}}$ be the (non complete) normal fan of P_v. Figure 1.2 is an example of the polyhedron P_v and of its normal fan Σ_{P_v} for the primitive vector $v = (2,3) \in \mathbb{Z}^2$. The fan Σ_{P_v} consists of the two 2-dimensional cones σ_1 and σ_2, the three rays spanned by the vectors $(1,0), (0,1)$ and $(2,3)$ and the 0-dimensional cone $(0,0)$.

Even though the fan Σ_{P_v} is not complete, it nevertheless corresponds to a smooth (non complete) toric variety $X_{\Sigma_{P_v}}$ and in the same manner as in the complete case, a regular subdivision $\Sigma'_{P_v} \geq \Sigma_{P_v}$ corresponds to a smooth (non complete) toric variety $X_{\Sigma'_{P_v}}$ together with a *proper* toric birational morphism $X_{\Sigma'_{P_v}} \to X_{\Sigma_{P_v}}$. We have the following lemma:

Lemma 1.4.1. *With the same notations as above, there exists a minimal smooth subdivision Σ'_{P_v} of Σ_{P_v}. Here, by minimal smooth subdivision we mean a smooth subdivision Σ'_{P_v} of Σ_{P_v} such that the induced toric morphism $X_{\Sigma'_{P_v}} \to X_{\Sigma_{P_v}}$ gives a* minimal resolution of singularities *as defined in [CLS10, Section 10].*

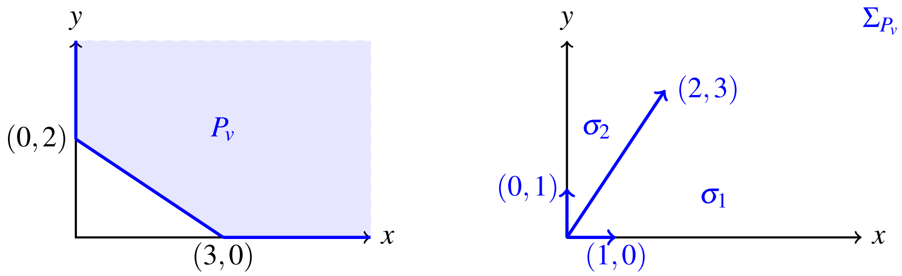


Figure 1.2: Picture of $P_{(2,3)}$ and its fan $\Sigma_{P_{(2,3)}}$

Proof. See [CLS10, Corollary 10.4.9]. ∎

Remark 1.4.2. We want to give an explicit description of the fan Σ'_{P_v} in the above theorem. Or at least, of the rays τ_{v_α} and τ_{v_β} which together with τ_v span a 2-dimensional cone in Σ'_{P_v}, i.e. such that $\text{cone}\langle v, v_\alpha\rangle$, $\text{cone}\langle v, v_\beta\rangle \in \Sigma'_{P_v}(2)$. The following lemma, whose proof follows from the computations done in the proof of [CLS10, Theorem 10.2.3] gives a formula for the vectors v_α and v_β.

Lemma 1.4.3. *Let $v = (v_1, v_2) \in N^{\text{prim}}$. Let x and y be integers uniquely defined by the following properties:*

(1) $y \leq 0$ *and* $x \geq 0$.

(2) $xv_1 + yv_2 = 1$.

(3) $0 \leq -y < v_1$.

(4) $0 \leq x < v_2$.

Then the primitive vectors v_α and v_β of Remark 1.3.13 are given by

$$v_\alpha = (-y, x) \quad \textit{and} \quad v_\beta = (v_1 + y, v_2 - x). \tag{1.4}$$

Remark 1.4.4. The construction of the polyhedral subdivision Σ'_{P_v} which can be found in the proof of [CLS10, Corollary 10.4.9/Theorem 10.2.3] is done by taking successive barycentric subdivisions of the 2-dimensional cones of Σ_{P_v}. It thus follows that the ray τ_v is obtained from the barycentric subdivision of $\text{cone}(v_\alpha, v_\beta)$. Indeed, this corresponds to the fact that $v = v_\alpha + v_\beta$, which follows from a straightforward computation using the formulas in (1.4).

Figure 1.3 shows the vectors v_α and v_β for the case $v = (2,3)$. Here we have $v_\alpha = (1,2)$ and $v_\beta = (1,1)$.

Let's return to toric b-divisors. Let $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ be a toric b-divisor on a smooth and complete toric surface X_Σ. We define a function

$$\mu_{\boldsymbol{D}} \colon N^{\text{prim}} \longrightarrow \mathbb{Q}$$

in the following way: let $v = (v_1, v_2) \in N^{\text{prim}}$. If $\tau_v \in \Sigma(1)$ then we put $\mu_{\boldsymbol{D}}(v) = 0$. Otherwise, let $\sigma \in \Sigma(2)$ be a 2-dimensional cone in Σ containing τ_v in its interior. Since Σ is smooth, we may assume that $\sigma = \text{cone}(e_1, e_2)$ is the positive quadrant. Consider the fan $\Sigma'_v = \Sigma \cup \Sigma'_{P_v}$, where

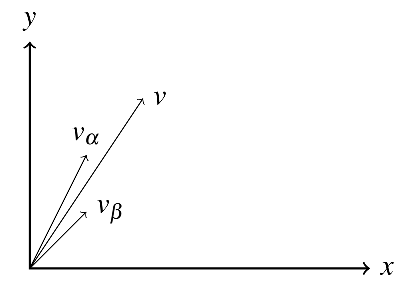


Figure 1.3: Picture for the case $v=(2,3)$, $v_\alpha=(1,2)$, $v_\beta=(1,1)$

Σ'_{P_v} is the normal fan of the Newton polyhedron P_v described previously. By construction, we have that Σ'_v is a smooth subdivision of Σ, i.e. it belongs to $R(\Sigma)$. Consider the toric birational morphism $\pi\colon X_{\Sigma'_v}\to X_{\Sigma'_v\setminus\{\tau_v\}}$. This is the blow up of $X_{\Sigma'_v\setminus\{\tau_v\}}$ at the torus fixed point corresponding to $\mathrm{cone}\langle v_\alpha,v_\beta\rangle\in\Sigma'_v\setminus\{\tau_v\}(2)$ with exceptional divisor E_{τ_v} corresponding to the ray τ_v. Define $\mu_{\boldsymbol{D}}(v)\in\mathbb{Q}$ to be the rational number satisfying

$$D_{\Sigma'_v}=\pi^*D_{\Sigma'_v\setminus\{\tau_v\}}+\mu_{\boldsymbol{D}}(v)E_{\tau_v}.$$

We call $\mu_{\boldsymbol{D}}$ the *jumping function* of the toric b-divisor $\boldsymbol{D}$.

Lemma 1.4.5. *Let $\tilde{\phi}$ be the conical function associated to the b-divisor $\boldsymbol{D}$ from Lemma 1.2.8. Then the function $\mu_{\boldsymbol{D}}\colon N^{\mathrm{prim}}\to\mathbb{Q}$ is given by the assignment*

$$v\longmapsto\tilde{\phi}(v)-(\tilde{\phi}(v_\alpha)+\tilde{\phi}(v_\beta)),$$

where v_α and v_β are the primitive vectors of Lemma 1.4.3.

Proof. Fix a primitive vector $v\in N^{\mathrm{prim}}$ and let $\sigma=\mathrm{cone}\langle v_\alpha,v_\beta\rangle$. Let ψ_σ be the linear function on σ determined by the values $\tilde{\phi}\left(v_\alpha\right)$ and $\tilde{\phi}\left(v_\beta\right)$. Then, the coefficient $\mu_{\boldsymbol{D}}(v)$ we are looking for is the difference $\tilde{\phi}(v)-\psi_\sigma(v)$. We compute

$$\tilde{\phi}(v)-\psi_\sigma(v)=\tilde{\phi}(v)-\left(\psi_\sigma\left(v_\alpha\right)+\psi_\sigma\left(v_\beta\right)\right)=\tilde{\phi}(v)-(\tilde{\phi}(v_\alpha)+\tilde{\phi}(v_\beta))$$

as claimed. ■

By definition, integrability of a toric b-divisor $\boldsymbol{D}$ is equivalent to the convergence of the sum $\sum_i\left(D^2_{\Sigma_{i+1}}-\pi^*_{i+1}D^2_{\Sigma_i}\right)$ of differences of toric degrees for every chain of blow ups

$$\cdots\xrightarrow{\pi_{n+1}}X_{\Sigma_n}\xrightarrow{\pi_n}\cdots\xrightarrow{\pi_1}X_{\Sigma_0}=X_\Sigma.$$

To simplify notation, let us write $a_i:=D^2_{\Sigma_{i+1}}-\pi^*_{i+1}D^2_{\Sigma_i}$. Note that we have canonical chains of blow ups consisting of taking successive barycentric subdivisions corresponding to blow ups of torus fixed points. Moreover, by Remark 1.2.12, the refinements $\Sigma'\geq\Sigma$ corresponding to barycentric subdivisions are cofinal in $R(\Sigma)$. Also, note that we have $(E_{\tau_v})^2=-1$. Hence, replacing the index set $\{i\in\mathbb{N}\}$ with the index set $\{v\in N^{\mathrm{prim}}\}$, we have that $a_v=-\mu(v)^2$. The following theorem follows from these considertaions together with Lemma 1.4.5.

Theorem 1.4.6. *Let notations be as above. A toric b-divisor $\boldsymbol{D}$ on a toric surface is integrable if and only if the infinite sum*

$$\sum_{v\in N^{prim}} \mu_{\boldsymbol{D}}(v)^2 = \sum_{\substack{v=(v_1,v_2)\in\mathbb{Z}^2\\ \gcd(v_1,v_2)=1}} \left(\tilde{\phi}(v)-\tilde{\phi}(v_\alpha)-\tilde{\phi}(v_\beta)\right)^2$$

is a finite number. Moreover, if this is the case, and if we denote the value of the series by μ, then the equality

$$\boldsymbol{D}^2 = D_\Sigma^2 - \mu$$

holds true.

Let's turn to an example.

Example 1.4.7. We choose and identification of lattices $N\simeq\mathbb{Z}^2$ and consider the fan $\Sigma\subseteq\mathbb{R}^2$ corresponding to the projective plane $\mathbb{P}^2$. Let $\boldsymbol{D}$ be the nef toric b-divisor given by the concave function

$$\phi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$

given by the assignment

$$(a,b)\longmapsto \begin{cases} \frac{ab}{a+b} & \text{if } \ a,b\geq 0, \text{ and } a+b>0,\\ \min\{a,b\} & \text{otherwise.}\end{cases}$$

Let $v=(v_1,v_2)$ be a pair of positive integers with $\gcd(v_1,v_2)=1$ and let x,y be integers satisfying the conditions in Lemma 1.4.3. Therefore, we have

$$v_\alpha = (-y,x) \quad \text{and} \quad v_\beta = (v_1+y, v_2-x).$$

We compute

$$\begin{aligned}\mu_{\boldsymbol{D}}(v) &= \phi(v)-\phi(v_\alpha)-\phi(v_\beta)\\ &= \frac{v_1v_2}{v_1+v_2}+\frac{xy}{x-y}-\frac{(v_1+y)(v_2-x)}{v_1+v_2-x+y}\\ &= \frac{(v_1x+v_2y)^2}{(v_1+v_2)(x-y)(v_1+v_2-x+y)}\\ &= \frac{1}{(v_1+v_2)(x-y)(v_1+v_2-x+y)}.\end{aligned}$$

Taking the sum over all coprime integers we get

$$\begin{aligned}\mu &= \sum_{\substack{v=(v_1,v_2)\in\mathbb{Z}^2\\ \gcd(v_1,v_2)=1}} \mu(v)^2 = \sum_{\substack{v=(v_1,v_2)\in\mathbb{Z}^2\\ \gcd(v_1,v_2)=1}} \left(\frac{1}{(v_1+v_2)(x-y)(v_1+v_2-x+y)}\right)^2\\ &= \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1}} \frac{1}{m^2n^2(m+n)^2} = \frac{1}{3}.\end{aligned}$$

For the third equality, we set $m=v_1+v_2$ and $n=x-y$ (note that $xm+v_2n=1$). For the fourth equality we refer to [BKK16, Section 4] and to [KP15, Section 3]. Combining the above calculation with Theorem 1.4.6 we get

$$\boldsymbol{D}^2 = D_{\mathbb{P}^2}^2 - \frac{1}{3} = H^2-\frac{1}{3} = 1-\frac{1}{3} = \frac{2}{3},$$

where H is the class of a hyperplane in $\mathbb{P}^2$.

Note that since $\boldsymbol{D}$ is nef, we can compute this degree also using the formula in Proposition 1.3.10. Now, to compute the stability set Δ_ϕ we proceed as follows: first, let us recall that Δ_ϕ is defined by

$$\Delta_\phi = \left\{(x,y) \in \left(\mathbb{R}^2\right)^\vee \,|\, xa+yb-\phi(a,b) \text{ is bounded below } \forall (a,b) \in \mathbb{R}^2\right\}$$

(see Section 1.1). The function ϕ is concave and it has two regions where it is linear. In these regions, the slope is given by $(1,0)$ and $(0,1)$. This gives us the vertices of Δ_ϕ and the line $(1,0)+t(0,1)$ for $t \in [0,1]$. In order to compute the non-linear part, we have to look at the tangent lines at ϕ in the positive orthant. We compute

$$\phi_a := \frac{\partial \phi}{\partial a} = \frac{b^2}{(a+b)^2} \quad \text{and} \quad \phi_b := \frac{\partial \phi}{\partial b} = \frac{a^2}{(a+b)^2}.$$

These are homogeneous functions of degree 0 and hence we can see them as functions of $t = a/b$. The points corresponding to the curved boundary of the convex set Δ_ϕ have coordinates given by $(x,y) = ((\phi_a(t), \phi_b(t))$ with $t \in (0,\infty)$. One sees that these points satisfy

$$\sqrt{x} + \sqrt{y} = 1.$$

Hence, we get the following description of the stability set of ϕ:

$$\Delta_\phi = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x,y \geq 0, \quad x+y \leq 1, \quad \sqrt{x}+\sqrt{y} \geq 1\right\}.$$

Finally, letting Δ be the standard simplex $\operatorname{convhull}\big((0,0),(1,0),(0,1)\big)$, we compute

$$\boldsymbol{D}^2 = 2!\operatorname{vol}\left(\Delta_\phi\right) = 2\operatorname{vol}(\Delta) - 2\int_0^1 \left(1-\sqrt{x}\right)^2 \mathrm{d}x = 1 - \frac{1}{3} = \frac{2}{3}.$$

Figure 1.4 relates the previous two calculations. Note that the $\mu_{\boldsymbol{D}}$-values (which are the numbers between the parenthesis depicted on the right) correspond to the volumes of the simplices which we successively subtract from the standard simplex Δ in order to approximate the volume of the convex set Δ_ϕ.

Remark 1.4.8. As is mentioned in the articles [BKK16] and [KP15], the number μ can be interpreted as the value of the *Mordell–Tornheim zeta function* evaluated at $(2,2;2)$. The latter paper gives a deeper insight into the connection between multiple zeta functions and arithmetic intersection numbers via volumes of simplices. This is a connection which should be further investigated in the future. In particular, intersection numbers of toric b-divisors in toric varieties of any dimension should be interpreted as multiple zeta values.

Let $\boldsymbol{D}$ be a toric b-divisor and let $\tilde{\phi}$ be its associated function on $N_\mathbb{Q}$. A natural question to ask is what are necessary and sufficient conditions on $\tilde{\phi}$ which ensure the integrability of $\boldsymbol{D}$. If $\tilde{\phi}$ does not extend to a continuous function ϕ on $N_\mathbb{R}$ there is not much we can say. The interesting case is if we assume that the extension exists. In this case, what properties must ϕ satisfy in order to ensure integrability? We have already seen that if ϕ is a difference of concave functions, then the b-divisor is integrable. But can we ask for less? Is the existence of an extension enough?

The answer to the last question is negative, as can be shown by Example 1.4.10. Before giving it, we introduce some notation.

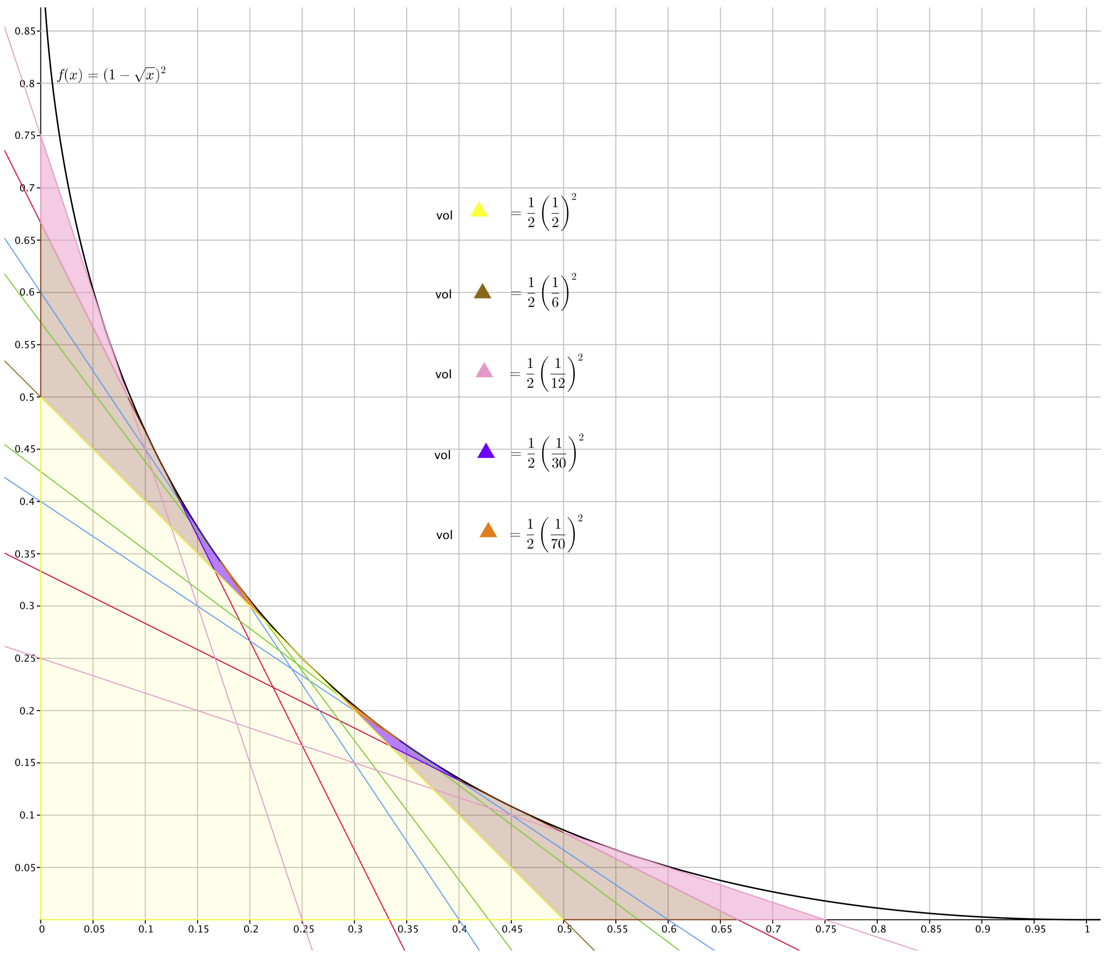


Figure 1.4: Cutting of simplices and μ-values

Definition 1.4.9. Let $f, g\colon \mathbb{N} \to \mathbb{R}$ be two $\mathbb{R}$-valued functions with domain the natural numbers $\mathbb{N}$. We say that f is equivalent to g (and write $f \sim g$), if there exist two positive real constants B, C such that $Bf(n) \leq g(n) \leq Cf(n)$ for all $n \gg 0$.

Now, we give an example of a non-integrable toric b-divisor for which the associated function $\tilde{\phi}$ on $\mathbb{N}_{\mathbb{Q}}$ extends to $\mathbb{N}_{\mathbb{R}}$. Note that this is actually a toric $\mathbb{R}$-b-divisor, not a toric $\mathbb{Q}$-b-divisor. The question whether we can find a non-integrable toric $\mathbb{Q}$-b-divisor whose corresponding function $\tilde{\phi}$ on $N_{\mathbb{Q}}$ extends continuously to $N_{\mathbb{R}}$ still remains open.

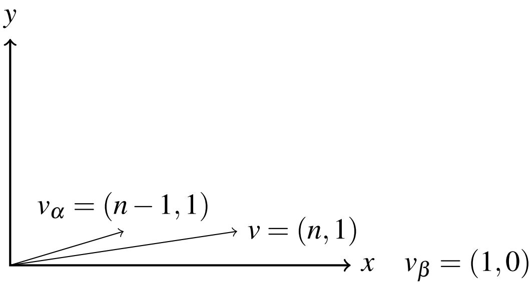
Example 1.4.10. Let $X_\Sigma = \mathbb{P}^2$. We choose an identification of lattices $N \simeq \mathbb{Z}^2$ and consider the toric b-divisor given by the conical function

$$\phi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$

defined by

$$(a,b) \longmapsto \begin{cases} \sqrt{|a||b|} & \text{if} \quad a,b \geq 0, \\ \min\{a,b\} & \text{otherwise.} \end{cases}$$

Note that the graph of this function mimics a cusp at the origin. Now, let n be any positive integer and consider the primitive vector $v = (n,1)$. We have $v_\alpha = (n-1,1)$ and $v_\beta = (1,0)$.



Using homogeneity we compute

$$\begin{aligned} \mu(v) &= \phi(v) - \phi(v_\alpha) - \phi(v_\beta) \\ &= n\phi\left(1, \frac{1}{n}\right) - (n-1)\phi\left(1, \frac{1}{n-1}\right) - \phi(1,0) \\ &= n\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}}\right) + \frac{1}{\sqrt{n-1}} \\ &\sim \frac{1}{\sqrt{n}}. \end{aligned}$$

Hence,

$$\sum_{\substack{v=(v_1,v_2)\in\mathbb{Z}^2 \\ (v_1,v_2)=1}} \mu(v)^2 \gg \sum_{n\geq 0} \frac{1}{n},$$

which diverges.

Remark 1.4.11. In exactly the same way as in the above example, one can show that a toric $\mathbb{R}$-b-divisor on $\mathbb{P}^2$ given by the conical function

$$\phi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$$

defined by

$$(a,b) \longmapsto \begin{cases} |a|^{\varepsilon}|b|^{1-\varepsilon} & \text{if} \quad a,b \geq 0, \\ \min\{a,b\} & \text{otherwise,} \end{cases}$$

where $\varepsilon \in (0,1)$, fails to be integrable.

We make the following remark.

Remark 1.4.12. We have seen in Lemma 1.3.14 that the difference of nef toric b-divisors is integrable. On the other hand, it is a classical result in algebraic geometry that every divisor on a smooth projective algebraic variety can be written as a difference of very ample divisors (in particular as the difference of nef divisors). The last example shows that this algebro-geometric fact does *not* extend to b-divisors.

1.5 b-Convex bodies and global sections of toric b-divisors

Throughout this section, $\Sigma \subseteq N_{\mathbb{R}}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. We start by classifying the class of convex bodies which arise from nef toric b-divisors. We then proceed to describe the space of global sections of a (not necessarily nef) toric b-divisor $\boldsymbol{D}$ in terms of lattice points in a more general convex set associated to the b-divisor, denoted by $\Delta_{\boldsymbol{D}}$. In the nef case, both convex sets $\Delta_{\boldsymbol{D}}$ and $\Delta_{\phi_{\boldsymbol{D}}}$ agree. In this case, we also show that a Hilbert–Samuel type formula holds relating the degree of the nef toric b-divisor with the asymptotic growth of the dimension of the space of global sections of multiples of the b-divisor. We also define the graded ring of global sections of multiples of a toric b-divisor and study the relationship between its finite generation and the polyhedrality of the associated convex set.

Convex bodies arising as a b-convex set

Recall from Theorem 1.3.10 that to a toric nef b-divisor $\boldsymbol{D}$ one can associate two objects: the concave extension $\phi_{\boldsymbol{D}} \colon N_{\mathbb{R}} \to \mathbb{R}$ and its stability set $\Delta_{\phi_{\boldsymbol{D}}} \subseteq M_{\mathbb{R}}$ which is a bounded convex body. A natural question to ask is the following: given a bounded convex set K in $M_{\mathbb{R}}$, does there exist a nef toric b-divisor $\boldsymbol{D}$ such that $K = \Delta_{\phi_{\boldsymbol{D}}}$? We answer this question in the following proposition.

Proposition 1.5.1. *The map given by the assignment*

$$\boldsymbol{D} \longmapsto \Delta_{\phi_{\boldsymbol{D}}}$$

induces a bijective correspondence between the set of nef toric b-divisors on X_Σ and the set of bounded convex bodies in $M_{\mathbb{R}}$ whose conical, concave support function $h \colon N_{\mathbb{R}} \to \mathbb{R}$ is rational, i.e. satisfies that $h(N_{\mathbb{Q}}) \subseteq \mathbb{Q}$.

Moreover, nef toric b-divisors on X_Σ which are Cartier correspond to rational polytopes whose normal fan belongs to $R(\Sigma)$. These have rational, piecewise linear support functions with regions of linearity given by the normal fan of the rational polytope.

Proof. Let $\boldsymbol{D}$ be a nef toric b-divisor. Then $\phi_{\boldsymbol{D}}(N_{\mathbb{Q}}) \subseteq \mathbb{Q}$ follows from the proof of Theorem 1.3.10 and the definition of $\phi_{\boldsymbol{D}}$. Conversely, given a bounded convex body K with rational support function h_K as in the statement of the proposition, Lemma 1.2.8 gives a toric b-divisor $\boldsymbol{D}$. Concavity of h_K implies that $\boldsymbol{D}$ is nef. These maps are clearly inverse to each other.
Now, if $\boldsymbol{D}$ is moreover Cartier, then $\boldsymbol{D}$ is determined on some toric model $X_{\Sigma'}$ for some $\Sigma' \in R(\Sigma)$.

Hence, $\phi_{\boldsymbol{D}} = \phi_{D_{\Sigma'}}$ satisfies $\phi_{D_{\Sigma'}}(N_{\mathbb{Q}}) \subseteq \mathbb{Q}$ and is piecewise linear with linearity regions given by Σ' and the stability set $\Delta_{\phi_{\boldsymbol{D}}} = \Delta_{\phi_{D_{\Sigma'}}}$ is a polytope with vertices in $M_{\mathbb{Q}}$. Conversely, starting with a rational polytope P whose normal fan Σ_P is in $R(\Sigma)$, its corresponding support function is piecewise linear with rational slopes and regions of linearity given by Σ_P. The corresponding Cartier nef toric b-divisor is the one determined by the toric divisor D_P on X_{Σ_P} associated to the rational polytope P. ■

Definition 1.5.2. A convex set which corresponds to a nef toric b-divisor as above is called a *b-convex set*.

Global sections of toric b-divisors

Definition 1.5.3. Let $D = \sum_{i=1}^{r} a_i D_i$ be a $\mathbb{Q}$-Weil divisor on an algebraic variety X. We define the $\mathbb{Z}$-Weil divisor $\lfloor D \rfloor$ by setting

$$\lfloor D \rfloor := \sum_{i=1}^{r} \lfloor a_i \rfloor D_i,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to a rational number x.

Definition 1.5.4. let D be a $\mathbb{Q}$-Weil divisor on a smooth algebraic variety X. Then its *space of global sections* is defined as

$$H^0(X, \mathcal{O}(D)) := H^0(X, \mathcal{O}(\lfloor D \rfloor)).$$

Now, let $D_\Sigma = \sum_{\tau \in \Sigma(1)} a_\tau D_\tau$ be a (not necessarily nef) toric $\mathbb{Q}$-Weil divisor on X_Σ and let $P_{D_\Sigma} \subseteq M_{\mathbb{R}}$ be its associated rational polyhedron. The proposition below is a classical result in toric geometry. However, since we are going to use similar ideas to generalize this to the b-setting, we give the proof also in this classical case.

Proposition 1.5.5. *With notations as above, the space of global sections of D_Σ is given by*

$$H^0(X, \mathcal{O}(D_\Sigma)) = \bigoplus_{m \in P_{D_\Sigma} \cap M} k \cdot \chi^m.$$

Proof. In order to simplify notation, we set $D = D_\Sigma$. The action of the torus $\mathbb{T}$ on itself given by multiplication induces an action of $\mathbb{T}$ on $k[M]$ as follows: if $t \in \mathbb{T}$ and $f \in k[M]$, then $t \cdot f \in k[M]$ is defined by $p \mapsto f(t^{-1} \cdot p)$. Then, by [CLS10, Section 4.3], the space $H^0(X_\Sigma, \mathcal{O}(\lfloor D \rfloor)) \subseteq k[M]$ is stable under this action and hence, by [CLS10, Lemma 1.1.16], it is given by

$$H^0(X_\Sigma, \mathcal{O}(\lfloor D \rfloor)) = \bigoplus_{\chi^m \in H^0(X_\Sigma, \mathcal{O}(\lfloor D \rfloor))} k \cdot \chi^m. \tag{1.5}$$

Using that the toric divisor corresponding to a character χ^m for $m \in M$ is given by

$$\operatorname{div}(\chi^m) = \sum_{\tau \in \Sigma(1)} \langle m, v_\tau \rangle D_\tau,$$

we conclude that

$$\begin{aligned} \chi^m \in H^0(X_\Sigma, \mathcal{O}(\lfloor D \rfloor)) \quad & \text{iff} \quad \operatorname{div}(\chi^m) \geq -\lfloor D \rfloor \\ & \text{iff} \quad \langle m, v_\tau \rangle \geq -\lfloor a_\tau \rfloor \geq -a_\tau \\ & \text{iff} \quad m \in P_{D_\Sigma}. \end{aligned} \tag{1.6}$$

Combining (1.5) and (1.6), the statement of the proposition follows. ■

Now, let $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ be a toric b-divisor which is not necessarily nef. We set

$$\lfloor \boldsymbol{D} \rfloor := (\lfloor D_{\Sigma'} \rfloor)_{\Sigma' \in R(\Sigma)}.$$

Also, note that a rational function $f \in k(X_\Sigma)^\times$ defines a b-divisor by setting

$$b\text{-div}(f) := (\operatorname{div}_{\Sigma'}(f))_{\Sigma' \in R(\Sigma)}.$$

We now define the space of global sections of a toric b-divisor.

Definition 1.5.6. The *space of global sections* $H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D}))$ of $\boldsymbol{D}$ is defined by

$$H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D})) := \left\{ f \in k(X_\Sigma) \,\middle|\, b\text{-div}(f) + \lfloor \boldsymbol{D} \rfloor \geq 0 \right\} \cup \{0\}.$$

The following lemma is a direct consequence of the definitions.

Lemma 1.5.7. *The space of global sections of a toric b-divisor $\boldsymbol{D}$ is given as the intersection of the spaces of global sections of the incarnations of $\boldsymbol{D}$ in the models $X_{\Sigma'}$ as Σ' varies in $R(\Sigma)$, i.e. the equality*

$$H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D})) = \bigcap_{\Sigma' \in R(\Sigma)} H^0\left(X_{\Sigma'}, \mathcal{O}(\lfloor D_{\Sigma'} \rfloor)\right)$$

is satisfied.

Definition 1.5.8. Let $\boldsymbol{D}$ be a toric b-divisor which is not necessarily nef and let $\tilde{\phi} \colon N_{\mathbb{Q}} \to \mathbb{Q}$ be its associated conical function. We define

$$\Delta_{\boldsymbol{D}} := \bigcap_{\Sigma' \in R(\Sigma)} P_{D_{\Sigma'}} = \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle m, v_\tau \rangle \geq -\tilde{\phi}(v_\tau),\ \forall v_\tau \in N^{\mathrm{prim}} \right\}.$$

This is a bounded convex set.

We make the following remarks.

Remark 1.5.9. (1) If $\boldsymbol{D}$ is nef, we have that $\Delta_{\boldsymbol{D}} = \Delta_{\phi_{\boldsymbol{D}}}$, where $\Delta_{\phi_{\boldsymbol{D}}}$ denotes the stability set of the concave extension $\phi_{\boldsymbol{D}} \colon N_{\mathbb{R}} \to \mathbb{R}$. In this case, we will use either notation interchangeably.
(2) In the non-nef case, $\Delta_{\boldsymbol{D}}$ does not necessarily correspond to a nef toric b-divisor as in Proposition 1.5.1. Indeed, any bounded convex body K in $M_{\mathbb{R}}$ is of the form $\Delta_{\boldsymbol{D}}$ for some toric b-divisor $\boldsymbol{D}$. This follows from the fact that any convex body is the intersection of a (possibly infinite) set of rational half spaces, the so called "rational supporting hyperplanes". Hence, while in the nef case we may recover the toric b-divisor $\boldsymbol{D}$ from the convex set $\Delta_{\phi_{\boldsymbol{D}}}$, in the non-nef case, we may not. This is also true in the classical case: a nef toric divisor D on a smooth and complete toric variety is completely determined by its corresponding polytope P_D, whereas in the non-nef case, the polytope does not capture all of the Cartier data of D. Nevertheless, the convex set $\Delta_{\boldsymbol{D}}$ gives us important algebro-geometric information. Like in the classical case, it encodes the global sections of the b-divisor, as can be seen by the following proposition.

Proposition 1.5.10. *Let $\boldsymbol{D}$ be a toric b-divisor on X_Σ. Its space of global sections is given by*

$$H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D})) = \bigoplus_{m \in \Delta_{\boldsymbol{D}} \cap M} k \cdot \chi^m.$$

Proof. By Lemma 1.5.7 and the proof of Proposition 1.5.5, the space $H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D})) \subseteq k[M]$ is $\mathbb{T}$-stable and can be written as

$$H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D})) = \bigoplus_{\chi^m \in H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D}))} k \cdot \chi^m.$$

We have

$$\begin{aligned}
\chi^m \in H^0(X_\Sigma, \mathcal{O}(\boldsymbol{D})) \quad &\text{iff} \quad \operatorname{div}_{\Sigma'}(\chi^m) \geq -\lfloor D_{\Sigma'} \rfloor, \forall \Sigma' \in R(\Sigma) \\
&\text{iff} \quad \langle m, v_\tau \rangle \geq -\lfloor \tilde{\phi}(v_\tau) \rfloor, \forall \tau \in \Sigma'(1) \text{ and } \forall \Sigma' \in R(\Sigma) \\
&\text{iff} \quad \langle m, v_\tau \rangle \geq -\lfloor \tilde{\phi}(v_\tau) \rfloor, \forall v_\tau \in N^{\text{prim}} \\
&\text{iff} \quad \langle m, v_\tau \rangle \geq -\tilde{\phi}(v_\tau), \forall v_\tau \in N^{\text{prim}} \\
&\text{iff} \quad m \in \Delta_{\boldsymbol{D}},
\end{aligned}$$

proving the statement of the proposition. ■

Hilbert–Samuel formula

The following is a Hilbert–Samuel type formula for toric nef b-divisors.

Theorem 1.5.11. *Let $\boldsymbol{D}$ be a nef toric b-divisor. We denote by $h^0(\ell\boldsymbol{D})$ the dimension of the space of global sections $H^0(X_\Sigma, \ell\boldsymbol{D})$ of an integral multiple $\ell\boldsymbol{D}$ of $\boldsymbol{D}$. Then we have a Hilbert–Samuel type formula*

$$\boldsymbol{D}^n = \lim_{\ell \to \infty} \frac{h^0(\ell\boldsymbol{D})}{\ell^n/n!}. \tag{1.7}$$

Proof. The so called *lattice volume* vol_L of a convex set $S \subseteq M_\mathbb{R}$ is defined by

$$\operatorname{vol}_L(S) := \lim_{\ell \to \infty} \frac{\#\ell S \cap M}{\ell^n}.$$

In the case where S is a polytope, we have $\operatorname{vol}(S) = \operatorname{vol}_L(S)$ (see e.g. [HKP06, Section 3]). Let ϕ be the concave function corresponding to $\boldsymbol{D}$. Then, since $\boldsymbol{D}$ is nef, using Proposition 1.5.10 we have that

$$\lim_{\Sigma' \in R(\Sigma)} h^0(X_{\Sigma'}, \ell D_{\Sigma'}) = h^0(X_\Sigma, \ell\boldsymbol{D}),$$

for all $\ell \in \mathbb{N}$. Moreover, the convex set $\Delta_{\boldsymbol{D}} = \Delta_\phi$ can be approximated by polytopes. Since the operator "vol" is continuous, the sequence of equalities

$$\begin{aligned}
\frac{\boldsymbol{D}^n}{n!} &= \operatorname{vol}(\Delta_\phi) = \lim_{\Sigma' \in R(\Sigma)} \operatorname{vol}(P_{D_{\Sigma'}}) \\
&= \lim_{\Sigma' \in R(\Sigma)} \lim_{\ell \to \infty} \frac{\#\ell P_{D_{\Sigma'}} \cap M}{\ell^n} \\
&= \lim_{\ell \to \infty} \lim_{\Sigma' \in R(\Sigma)} \frac{\#\ell P_{D_{\Sigma'}} \cap M}{\ell^n} \\
&= \lim_{\ell \to \infty} \frac{\#\ell \Delta_\phi \cap M}{\ell^n}
\end{aligned}$$

is satisfied, thus concluding the proof of the Theorem. ■

Ring of global sections of multiples of a toric b-divisor

As in the classical case, a natural object to study is the ring of global sections of multiples of a toric b-divisor. In order to define this, note that given two toric b-divisors $\boldsymbol{D}$ and $\boldsymbol{E}$ together with two rational functions f in $H^0(X_\Sigma, \boldsymbol{D})$ and g in $H^0(X_\Sigma, \boldsymbol{E})$, it is easy to see that the product $f \cdot g$ is an element in $H^0(X_\Sigma, \boldsymbol{D}+\boldsymbol{E})$. Hence, we get a map

$$H^0(X_\Sigma, \boldsymbol{D}) \otimes H^0(X_\Sigma, \boldsymbol{E}) \longrightarrow H^0(X_\Sigma, \boldsymbol{D}+\boldsymbol{E}).$$

Definition 1.5.12. Let $\boldsymbol{D}$ be a nef toric b-divisor on X_Σ. We define the *ring of global sections of multiples of* $\boldsymbol{D}$ as the graded ring

$$b\text{-}R(\boldsymbol{D}) := \bigoplus_{\ell \geq 0} H^0\left(X_\Sigma, \mathcal{O}(\ell \boldsymbol{D})\right).$$

It is a natural question to ask whether b-$R(\boldsymbol{D})$ is finitely generated as a graded k-algebra. We answer this question in the following proposition.

Proposition 1.5.13. *Let $\boldsymbol{D}$ be a nef toric b-divisor and let $\phi \colon N_\mathbb{R} \to \mathbb{R}$ be its corresponding concave function. Then the ring b-$R(\boldsymbol{D})$ is finitely generated if and only if $\boldsymbol{D}$ is Cartier, i.e. if Δ_ϕ is a rational polytope.*

Proof. Let $\boldsymbol{D}$ be a Cartier toric b-divisor. By Proposition 1.5.1, the corresponding convex set $\Delta_{\boldsymbol{D}} = \Delta_{\phi_{\boldsymbol{D}}}$ is a rational polytope and it corresponds to the polytope $P_{D_{\Sigma''}}$ of a Cartier toric $\mathbb{Q}$-divisor $D_{\Sigma''}$ on a toric model $X_{\Sigma''}$ for some fan $\Sigma'' \in R(\Sigma)$. Clearly, in this case, the ring b-$R(\boldsymbol{D})$ is finitely generated if and only if $R(D_{\Sigma''})$ is finitely generated and the finite generatedness of the latter object is a classical result (see e.g. [Eli97]).
To see the "only if" part, suppose that b-$R(\boldsymbol{D})$ is finitely generated. Let $m_1, \ldots, m_r$ be elements in the lattice M be such that χ^{m_i} generate b-$R(\boldsymbol{D})$ for $i = 1, \ldots, r$. Let $\tilde{\Sigma} \geq \Sigma$ be sufficiently large so that $\chi^{m_i} \in R(D_{\tilde{\Sigma}})$ for all $i = 1, \ldots, r$. Then, for some multiple $\ell \in \mathbb{N}$, we have

$$P = \ell P_{D_{\tilde{\Sigma}}},$$

where $P = \operatorname{convhull}(m_1, \ldots, m_r)$. Hence, the normal fan Σ_P of P is a smooth subdivision of Σ, i.e. $\Sigma_P \in R(\Sigma)$. Moreover, for every $\Sigma' \geq \Sigma_P$, we have

$$\ell P_{D_{\Sigma'}} = \ell P_{D_{\tilde{\Sigma}}} = P.$$

Indeed, $P_{D_{\Sigma'}}$ is the convex hull of the generators of global sections. Hence, since we have that $\Delta_\phi = \lim_{\Sigma'} P_{D_\Sigma} = \frac{1}{\ell} P$, the statement of the proposition follows. ■

Ehrhart function

We now come to the Ehrhart function of convex sets, which has been widely studied in the polyhedral case.

Definition 1.5.14. For a bounded convex set $K \subseteq M_\mathbb{R}$ the *Ehrhart function of* K is the function $f_K \colon \mathbb{N} \to \mathbb{N}$ defined by

$$f_K(\ell) := \# \ell K \cap M,$$

i.e. the number of lattice points in the dilated convex set $\ell K := \{\ell p \,|\, p \in K\}$.

If $K = P$ is a rational polytope, let $t \in \mathbb{N}$ be a positive integer such that the vertices of the dilated polytope $tP = \{tx \,|\, x \in P\}$ are in the lattice M. Then, as is known (see for example [Sta97, Section 4.6]), there exist functions $e_i(P;\cdot)\colon \mathbb{N} \to \mathbb{Q}$, for $i = 0,\dots,n$, such that the following holds:

$$e_i(P;\ell+t) = e_i(P;\ell),\, \forall \ell \in \mathbb{N},$$
$$\#\ell P \cap M = \sum_{i=0}^{n} e_i(P;\ell)\ell^i,\, \forall \ell \in \mathbb{N}.$$

The function given by the assignment

$$\ell \longmapsto \sum_{i=0}^{n} e_i(P;\ell)\ell^i$$

is called the *Ehrhart quasi-polynomial* of P. Recall that a quasi-polynomial is like a polynomial but the coefficients are periodic functions with integral period. Note that if P is full-dimensional then $e_n(P;\ell) = \mathrm{vol}(P)$.
A natural question is whether there is a converse to the above result, i.e. if f_C is a quasi-polynomial, is it true that C is a rational polytope?
The answer to the above question is negative, as there are polytopes with irrational vertices whose Ehrhart function is even a polynomial (see [CGLS15]). However, we can formulate the following question.

Question 1.5.15. Is it true that if $K = \Delta_{\boldsymbol{D}} \subseteq M_{\mathbb{R}}$ is a full-dimensional convex body associated to a nef toric b-divisor $\boldsymbol{D}$ which is not Cartier, then f_K is not a polynomial? Equivalently, by Proposition 1.5.13, is it true that the Hilbert function of a *not* finitely generated graded ring is never a polynomial?

The b-Cox ring

A further interesting object to study is the b-Cox ring of a complete, smooth toric variety. Before giving the definition, we recall the notion of the classical Cox ring of a toric variety.

Definition 1.5.16. The *Cox ring* of a toric variety X_Σ is the polynomial ring

$$\mathrm{Cox}(X_\Sigma) := k\left[\left\{x_\tau \,\middle|\, \tau \in \Sigma(1)\right\}\right].$$

This ring is multigraded by the class group $\mathrm{Cl}(X_\Sigma)$ of X_Σ. Indeed, we have the following short exact sequence:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where $\mathbb{Z}^{\Sigma(1)}$ is the free group generated by symbols y_τ for $\tau \in \Sigma(1)$. The first morphism sends $m \in M$ to $\mathrm{div}(\chi^m) = \sum_\tau \langle m, v_\tau \rangle y_\tau$ and the second one sends y_τ to the class $[D_\tau] \in \mathrm{Cl}(X_\Sigma)$ of the corresponding toric divisor. The degree of y_τ is just $[D_\tau]$.

We now define the *b-Cox ring* and the *b-class group* of X_Σ.

Definition 1.5.17. The *b-Cox ring* of X_Σ is the following polynomial ring in infinitely many variables:

$$\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right) := k\left[\left\{x_v \,\middle|\, v \in N^{\mathrm{prim}}\right\}\right].$$

Definition 1.5.18. The *b-class group* of X_Σ is defined as the inverse limit of class groups of toric varieties

$$\mathrm{Cl}\left(\mathfrak{X}_\Sigma\right) := \varprojlim_{\Sigma' \in R(\Sigma)} \mathrm{Cl}\left(X_{\Sigma'}\right).$$

with maps given by the push-forward map of divisor classes of Weil toric divisors. The class of a toric b-divisor $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ in $\mathrm{Cl}\left(\mathfrak{X}_\Sigma\right)$ is defined by

$$[\boldsymbol{D}] := \left(\left[\lfloor D_{\Sigma'} \rfloor\right]\right)_{\Sigma' \in R(\Sigma)}.$$

Note that we are taking only integer coefficients. As in the classical case, we have a natural grading on $\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right)$ by the b-class group $\mathrm{Cl}\left(\mathfrak{X}_\Sigma\right)$. Indeed, we have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{N^{\mathrm{prim}}} \longrightarrow \mathrm{Cl}\left(\mathfrak{X}_\Sigma\right) \longrightarrow 0$$

where, as before, the first morphism is given by the assignment

$$m \longmapsto \left(\langle m, v \rangle\right)_{v \in N^{\mathrm{prim}}},$$

for $m \in M$, and the second morphsim by the assignment

$$(m_v)_{v \in N^{\mathrm{prim}}} \longmapsto \left[\sum_v m_v D_v\right] = \left(\left[\sum_{\tau \in \Sigma'(1)} m_{\tau_v} D_{\tau_v}\right]\right)_{\Sigma' \in R(\Sigma)}.$$

Given $a \in N^{\mathrm{prim}}$, an element $x^a = \prod_v x_v^{a_v}$ in $\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right)$, the degree of x^a is defined by

$$\deg(x^a) := \left[\sum_v a_v D_v\right] = \left(\left[\sum_{\tau \in \Sigma'(1)} a_{v_\tau} D_{v_\tau}\right]\right)_{\Sigma' \in R(\Sigma)} \in \mathrm{Cl}\left(\mathfrak{X}_\Sigma\right).$$

Generalizing the result in [LV09, Theorem 1.4] for the case of b-divisors, we have the following proposition.

Proposition 1.5.19. *The vector space $H^0(X_\Sigma, \boldsymbol{D})$ of global sections of $\boldsymbol{D}$ is isomorphic to the degree $[\boldsymbol{D}]$ part of* $\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right)$.

Proof. Let $\boldsymbol{D}$ be a toric b-divisor given by $\boldsymbol{D} = \sum_{i \in \mathbb{N}} a_{v_i} D_{v_i}$. A rational function χ^m ($m \in M$) is in $H^0(X_\Sigma, \boldsymbol{D})$ if and only if $\mathrm{div}(\chi^m) + \lfloor \boldsymbol{D} \rfloor \geq 0$, or equivalently, if $\langle m, v_i \rangle \geq -\lfloor a_{v_i} \rfloor$ for all $i \in \mathbb{N}$. Hence, the expression

$$x_1^{\langle m, v_1 \rangle + \lfloor a_{v_1} \rfloor} \cdots x_d^{\langle m, v_d \rangle + \lfloor a_{v_d} \rfloor} \cdots$$

has only nonnegative exponents and this defines a monomial of degree $[\boldsymbol{D}]$ in $\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right)$. Conversely, let $x^s = \left(x_{v_i}^{s_{v_i}}\right)_{i \in \mathbb{N}}$ be a monomial of degree $[\boldsymbol{D}]$ in $\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right)$. There exists an $m \in M$ such that

$$\sum_i s_{v_i} D_{v_i} = \sum_i a_{v_i} D_{v_i} + \mathrm{div}\left(\chi^m\right).$$

Hence,

$$\mathrm{div}\left(\chi^m\right) + \lfloor \boldsymbol{D} \rfloor = \langle m, v_i \rangle + \lfloor a_{v_i} \rfloor = s_{v_i} \geq 0,$$

which implies that $m \in H^0(X_\Sigma, \boldsymbol{D})$. This correspondence is obviously bijective. ■

Remark 1.5.20. It is a classical result (see e.g [CLS10, Theorem 12.5.3]) that the Chow ring $\mathrm{A}^{\bullet}_{\mathbb{Q}}(X_\Sigma)$ of a toric variety X_Σ is a quotient of the Cox ring. Specifically, we have the following:

$$\mathrm{A}^{\bullet}_{\mathbb{Q}}(X_\Sigma) = \mathrm{Cox}(X_\Sigma)/(\mathcal{I}+\mathcal{J}),$$

where $\mathcal{I}$ is the monomial ideal with square-free generators

$$\mathcal{I} := \left\langle x_{i_1}\cdots x_{i_s} \,\middle|\, i_j \text{ are distinct and } \mathrm{cone}(\tau_{i_1},\dots,\tau_{i_s}) \text{ is not a cone of } \Sigma \right\rangle$$

and $\mathcal{J}$ is the ideal generated by the linear forms

$$\mathcal{J} := \left\langle \sum_{\tau\in\Sigma(1)} \langle m, v_\tau\rangle x_\tau \,\middle|\, m \in M \right\rangle.$$

The b-Cox ring of a complete toric variety is thus the natural object where one hopes to develop a suitable intersection theory of toric b-divisors other than just for top degrees.

Chapter 2

Relationship with Okounkov bodies

Okounkov bodies (in the literature often called Newton–Okounkov bodies) are convex bodies which one can attach to an algebraic variety together with some extra data, e.g. a divisor. These convex bodies have been widely and successfully used to explore the geometry of the variety using convex geometrical methods (see [Oko96, Oko03] and also [KK12, KK14, KK08] and [LM09] and the references therein). The main goal of this chapter is to identify the convex sets $\Delta_{\boldsymbol{D}}$ arising from toric b-divisors $\boldsymbol{D}$ (Definition 1.5.8) with convex Okounkov bodies attached to algebras of almost integral type defined by K. Kaveh and A. G. Khovanskii in [KK12]. As an application, in the big and nef case, we also construct a global Okounkov body generalizing the global Okounkov body associated to a big divisor constructed in [LM09] to the b-setting.

2.1 Identification of b-convex bodies and Okounkov bodies

Throughout this section, $\Sigma \subseteq N_{\mathbb{R}}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. A *convex body* is a compact, convex set in $\mathbb{R}^n$ with non-empty interior. We start by recalling the definition of the convex Okounkov body associated to an algebra of almost integral type defined in [KK12] (where it is called the Newton–Okounkov body). Later on, we will apply this construction to the case of the ring of global sections of multiples of a (not necessarily nef) toric b-divisor. This turns out to be an algebra of almost integral type. Then we give an isomorphism of lattices $M \simeq \mathbb{Z}^n$ which identifies the convex set $\Delta_{\boldsymbol{D}}$ of Definition 1.5.8 with the associated Okounkov body.

Okounkov body associated to an algebra of almost integral type

We start with some definitions which are taken from [KK12, Section 2.3].

Definition 2.1.1. Let F be a finitely generated field of transcendence degree n over k. For any integer $\ell \geq 0$, a homogeneous element of degree ℓ in $F[t]$ is an element $a_\ell t^\ell$ where $a_\ell \in F$. Let B be a k-linear subspace of $F[t]$. The collection B_ℓ of homogeneous elements of degree ℓ in B is a k-linear subspace of B called the ℓ-th homogeneous component of B. The k-linear space $L_\ell \subseteq F$ such that $a \in L_\ell$ if and only if $at^\ell \in B_\ell$ is called the ℓ-th homogeneous subspace of B. A k-linear subspace $B \subseteq F[t]$ is called *graded* if it is the direct sum of its homogeneous components. A subalgebra $A \subseteq F[t]$ is called *graded* if it is graded as a linear subspace of $F[t]$.

Definition 2.1.2. We define the following three classes of graded subalgebras of $F[t]$.

(1) To each non-zero finite dimensional linear subspace $L \subseteq F$ over k, we associate the graded subalgebra $A_L \subseteq F[t]$ defined as follows: it's zeroth homogeneous component is k and for each $\ell > 0$, its ℓ-th subspace is L^ℓ, the subspace spanned by all products $f_1 \cdots f_\ell$ with $f_1, \dots, f_\ell \in L$. That is, $A_L = \bigoplus_{\ell \geq 0} L^\ell t^\ell$.

(2) A graded subalgebra $A \subseteq F[t]$ is called an *algebra of integral type* if there is a graded subalgebra $A_L \subseteq F[t|$ for some non-zero finite dimensional linear subspace $L \subseteq F$ over k such that the subalgebra A is a finitely generated A_L-module (equivalently, A is finitely generated over k and is a finite module over the subalgebra generated by A_1, the first homogeneous components of A).

(3) Finally, a graded subalgebra $A \subseteq F[t]$ is called an *algebra of almost integral type* if there is an algebra $A' \subseteq F[t]$ of integral type such that $A \subseteq A'$ (equivalently, $A \subseteq A_L$ for some non-zero finite dimensional linear subspace $L \subseteq F$).

Remark 2.1.3. We can think of the A_L's as being the homogeneous coordinate rings of projective varieties. Then, one can see that rings of global sections of ample line bundles give rise to algebras A of integral type, whereas rings of global sections of arbitrary divisors give rise to algebras of almost integral type ([KK12, Theorems 3.7 and 3.8]). It turns out that also toric b-divisors give rise to algebras of the latter type.

Definition 2.1.4. Let A be an algebra over k and let G be an ordered abelian group, i.e. an abelian group equipped with a total order "$<$" respecting the group operation. A *valuation* on A is a function

$$\nu \colon A \setminus \{0\} \longrightarrow G$$

satisfying the following conditions:

(1) For all $f, g \in A$ with $f, g, f+g \neq 0$, we have $\nu(f+g) \geq \min(\nu(f), \nu(g))$.

(2) For all $0 \neq f \in A$ and $0 \neq \lambda \in k$, we have $\nu(\lambda f) = \nu(f)$.

(3) For any $f, g \in A$ with $f, g \neq 0$, we have $\nu(fg) = \nu(f) + \nu(g)$.

A valuation ν is said to be *faithful* if its image is the whole of G. It is said to *have* 1*-dimensional leaves* if it satisfies that if whenever $\nu(f) = \nu(g)$ for $f, g \in F \setminus \{0\}$, then

$$\nu(g + \lambda f) > \nu(g)$$

for some $\lambda \in k$.

We associate a semigroup $S(A) \subseteq \mathbb{Z}^n \times \mathbb{Z}$ to an algebra $A \subseteq F[t]$ of almost integral type. This is done as follows: first, assume we are given a faithful valuation

$$\nu \colon F \setminus \{0\} \longrightarrow \mathbb{Z}^n$$

with 1-dimensional leaves (we assume $\mathbb{Z}^n$ to be equipped with a total order "$<$" which respects addition). We now consider the total ordering $\prec$ on the group $\mathbb{Z}^n \times \mathbb{Z}$ given in the following way: let (α, i), $(\beta, j) \in \mathbb{Z}^n \times \mathbb{Z}$.

(1) If $i < j$, then $(\alpha, i) \prec (\beta, j)$.

(2) If $i = j$ and $\alpha < \beta$, then $(\alpha, i) \prec (\beta, j)$.

Definition 2.1.5. Given a valuation ν on F as above, we define a valuation

$$\nu_t\colon F[t] \setminus \{0\} \longrightarrow \mathbb{Z}^n \times \mathbb{Z}$$

in the following way: let $p(t) = a_i t^i + \cdots + a_0$, with $a_i \neq 0$, be a polynomial in $F[t]$. Then we put

$$\nu_t(p) := (\nu(a_i), i)$$

and extend ν_t to $F(t) \setminus \{0\}$. This is a faithful valuation with 1-dimensional leaves extending ν on F which we also denote by ν_t.

We are now ready to define the semigroup $S(A) \subseteq \mathbb{Z}^n \times \mathbb{Z}$ associated to an algebra $A \subseteq F[t]$ of almost integral type.

Definition 2.1.6. Let $A \subseteq F[t]$ be an algebra of almost integral type and assume that we are given a faithful valuation ν on F with 1-dimensional leaves. We define the semigroup $S(A)_\nu$ by

$$S(A)_\nu := \nu_t(A \setminus \{0\}) \subseteq \mathbb{Z}^n \times \mathbb{Z}.$$

We sometimes write $S(A) = S(A)_\nu$ keeping in mind the dependence on the valuation ν. This semigroup has some nice properties.

Definition 2.1.7. A pair (S, R) where S is a semigroup in $\mathbb{Z}^n$ and R is a rational half-space in

$$L(S) := \text{real span of } S \subseteq \mathbb{R}^n$$

is said to be *admissible* if $S \subseteq R$. An admissible pair is called *strongly admissible* if, moreover, the cone over S defined by

$$\operatorname{cone}(S) := \overline{\operatorname{convhull}(S \cup \{0\})} \subseteq \mathbb{R}^n \tag{2.1}$$

is strictly convex and intersects the boundary ∂R of R only at the origin.

Definition 2.1.8. A *non-negative* semigroup of integral points in $\mathbb{R}^{n+1}$ is a semigroup

$$S \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq 0}$$

which is not contained in the hyperplane $x_{n+1} = 0$. To such a semigroup one can associate an admissible pair $(S, R(S))$ by setting

$$R(S) := L(S) \cap (\mathbb{R}^n \times \mathbb{R}_{\geq 0}).$$

A non-negative semigroup S is said to be strongly admissible if its associated admissible pair $(S, R(S))$ has this property.

The following is [KK12, Theorem 2.31].

Lemma 2.1.9. *The semigroup $S(A) \subseteq \mathbb{Z}^{n+1}$ of Definition 2.1.6 is non-negative and its associated admissible pair $(S(A), R(S(A)))$ is strongly admissible.*

We are now ready to define the convex Okounkov body associated to an algebra of almost integral type.

Definition 2.1.10. Let $A \subseteq F[t]$ be an algebra of almost integral type and let $S(A)$ be the non-negative, strongly admissible semigroup defined above. Consider the strongly convex cone C given by

$$C = \operatorname{cone}(S(A)) \subseteq \mathbb{R}^n \times \mathbb{R}.$$

The *Okounkov body* Δ_A associated to A is then defined to be the slice of C at height 1, i.e.

$$\Delta_A := C \cap (\mathbb{R}^n \times \{1\}).$$

Okounkov body associated to the ring of global sections of multiples of a toric *b*-divisor

We now proceed to show that the ring of global sections of multiples of a toric b-divisor $\boldsymbol{D}$ defines an algebra of almost integral type $A_{\boldsymbol{D}}$ and that its associated Okounkov body $\Delta_{A_{\boldsymbol{D}}}$ can be identified with the convex set $\Delta_{\boldsymbol{D}}$ associated to $\boldsymbol{D}$ from Definition 1.5.8.

Let $F = k(X_\Sigma)$ be the field of rational functions of the complete, smooth toric variety X_Σ. This is the quotient field of $k[M]$, the ring of Laurent polynomials. Hence, elements of F are quotients of elements of the form $\sum_{m\in M} a_m\chi^m$, where, as usual, χ^m denotes the character of the torus of weight $m \in M$. Recall from the previous chapter that to a toric b-divisor $\boldsymbol{D}$ one can associate the space of global sections $H^0(X_\Sigma,\boldsymbol{D}) \subseteq k[M]$ which is given by

$$H^0(X_\Sigma,\boldsymbol{D}) = \{f \in F \,|\, b\text{-div}(f) + \lfloor \boldsymbol{D} \rfloor \geq 0\}.$$

Moreover, we have a well defined map

$$H^0(X_\Sigma,\boldsymbol{D}) \otimes H^0(X_\Sigma,\boldsymbol{E}) \longrightarrow H^0(X_\Sigma,\boldsymbol{D}+\boldsymbol{E}) \tag{2.2}$$

for any toric b-divisors $\boldsymbol{D}$ and $\boldsymbol{E}$.

Definition 2.1.11. We define the set $A_{\boldsymbol{D}}$ to be the collection of all polynomials $f(t) = \sum_\ell f_\ell t^\ell$ with $f_\ell \in H^0(X_\Sigma, \ell\boldsymbol{D})$, i.e.

$$A_{\boldsymbol{D}} = \bigoplus_{\ell\geq 0} H^0(X_\Sigma, \ell\boldsymbol{D})t^\ell.$$

By (2.2), $A_{\boldsymbol{D}} \subseteq F[t]$ is a graded subalgebra.

Remark 2.1.12. Note that the graded subalgebra $A_{\boldsymbol{D}}$ is like the graded algebra b-$R(\boldsymbol{D})$ from Definition 1.5.12, with the difference that in $A_{\boldsymbol{D}}$ we are taking track of the grading with the variable t. We choose to use this notation in order to be compatible with the notation in [KK12].

Proposition 2.1.13. *Let $\boldsymbol{D}$ be a toric b-divisor on X_Σ. Then the graded subalgebra $A_{\boldsymbol{D}} \subseteq F[t]$ is an algebra of almost integral type.*

Proof. This follows from the inclusion

$$A_{\boldsymbol{D}} \subseteq A_{D_{\Sigma'}}$$

of algebras for any fan $\Sigma' \in R(\Sigma)$ and the fact that $A_{D_{\Sigma'}} \subseteq F[t]$ is finitely generated and hence an algebra of integral type (see [Eli97]). ■

We want to describe the Okounov body $\Delta_{A_{\boldsymbol{D}}}$ associated to the algebra of integral type $A_{\boldsymbol{D}}$. Recall from the previous section that the convex body $\Delta_{A_{\boldsymbol{D}}}$ depends on the choice of a faithful valuation $\nu\colon F\setminus\{0\} \to \mathbb{Z}^n$ with 1-dimensional leaves. Our goal is to give an identification of lattices $\phi\colon M \to \mathbb{Z}^n$ and a faithful valuation $\nu\colon F\setminus\{0\} \to \mathbb{Z}^n$ with 1-dimensional leaves such that, if we denote by $\phi_{\mathbb{R}}\colon M_{\mathbb{R}} \to \mathbb{R}^n$ the induced isomorphism of real vector spaces, then we get an identification

$$\Delta_{A_{\boldsymbol{D}}} = \phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}})$$

of convex bodies.

In order to do this, we follow a construction presented in [LM09, Section 6.1]. First, let us start with a complete flag

$$Y_\bullet = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{\mathrm{pt}\}$$

of $\mathbb{T}$-invariant subvarieties of X_Σ. We have $\operatorname{codim}_{X_\Sigma}(Y_i) = i$ for $i = 0,\dots,n$. Since we are assuming X_Σ to be smooth, we can order the prime $\mathbb{T}$-invariant divisors $D_1,\dots,D_r$ of X_Σ, where $r = \#\Sigma(1)$, in such a way that we have $Y_i = D_1 \cap \cdots \cap D_i$ for $i \leq n \leq r$.

For $i = 1,\dots,r$, we denote by v_i the primitive generator of the ray corresponding to D_i. Then the set of vectors $\{v_1,\dots,v_n\}$ forms a basis of N and generates a cone σ in Σ of maximal dimension n. We get an isomorphism

$$\phi\colon M \simeq \mathbb{Z}^n, \tag{2.3}$$

given by

$$u \longmapsto (\langle u, v_i\rangle)_{1\leq i\leq n}.$$

We denote by $\phi_{\mathbb{R}}\colon M_{\mathbb{R}} \simeq \mathbb{R}^n$ the induced isomorphism of real vector spaces.

Definition 2.1.14. Let F and $Y_\bullet$ be as above. We define the valuation $\nu = \nu_{Y_\bullet}\colon F\setminus\{0\} \to \mathbb{Z}^n$ in the following way: let $s \in k[M]$ and let $\operatorname{div}(s) = \sum_{i=1}^r a_i D_i$ be the zero locus of s. Then $\nu(s)$ is defined to be the tuple $(a_1,\dots,a_n)$. Finally, we extend by linearity to $F\setminus\{0\}$.

One can check that

$$\#\mathrm{Im}\left(\nu\colon H^0(X_\Sigma, \ell\boldsymbol{D})\setminus\{0\} \longrightarrow \mathbb{Z}^n\right) = h^0(X_\Sigma, \ell\boldsymbol{D})$$

for any integer multiple $\ell \geq 0$. Moreover, by [KK12], the valuation ν is faithful and has 1-dimensional leaves. We can now state the main theorem of this section.

Theorem 2.1.15. *Let notations be as above and consider a toric b-divisor $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma'\in R(\Sigma)}$ such that $D_\Sigma|_{U_\sigma}$ is trivial. Let $A_{\boldsymbol{D}} \subseteq F[t]$ be the algebra of almost integral type associated to $\boldsymbol{D}$ from Definition 2.1.11. Let $\nu\colon F\setminus\{0\} \to \mathbb{Z}^n$ be the valuation of Definition 2.1.14 and let $\phi_{\mathbb{R}}$ be the isomorphism (2.3). Then we can identify the convex sets $\Delta_{A_{\boldsymbol{D}}} = \Delta^{\nu}_{A_{\boldsymbol{D}}}$ and $\Delta_{\boldsymbol{D}}$, i.e. we have that*

$$\Delta_{A_{\boldsymbol{D}}} = \phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}}).$$

Proof. Recall from the previous section that the semigroup $S(A_{\boldsymbol{D}})$ is defined to be the image $\nu_t(A_{\boldsymbol{D}}\setminus\{0\}) \subseteq \mathbb{Z}^n\times\mathbb{Z}$. Also, recall that the strictly convex cone $C \subseteq \mathbb{R}^n\times\mathbb{R}$ is given by

$$C = \operatorname{cone}(S(A_{\boldsymbol{D}})) = \overline{\operatorname{convhull}\left(S(A_{\boldsymbol{D}})\cup\{0\}\right)}.$$

Consider the convex body $\phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}}) \subseteq \mathbb{R}^n\times\mathbb{R}$ lying in the slice $\mathbb{R}^n\times\{1\}$. We define the cone $C' \subseteq \mathbb{R}^n\times\mathbb{R}$ by

$$C' := \text{strictly convex cone in } \mathbb{R}^n\times\mathbb{R} \text{ over the convex set } \phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}}).$$

Note that we have $\phi_{\mathbb{R}}(\Delta_{\ell\boldsymbol{D}}) = \phi_{\mathbb{R}}(\ell\Delta_{\boldsymbol{D}}) = \ell\phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}})$. Hence, C' is characterized by

$$C'\cap(\mathbb{R}^n\times\{\ell\}) = \phi_{\mathbb{R}}(\Delta_{\ell\boldsymbol{D}}).$$

In order to prove the theorem it suffices to show that $C = C'$. Moreover, since C (resp. C') is the convex hull of its lattice points, it suffices to show that C and C' contain the same lattice points. Indeed, let $(\phi(m),\ell) \in C'\cap(\mathbb{Z}^n\times\mathbb{Z})$ so that $m \in \Delta_{\ell\boldsymbol{D}}$. The zero locus of the corresponding section $\chi^m \in H^0(X_\Sigma, \ell\boldsymbol{D})$ is $\ell\boldsymbol{D} + \sum_{i=1}^r \langle m, v_i\rangle D_i$. By assumption, we have that the coefficient of D_i in $\boldsymbol{D}$ is 0 for $i = 1,\dots,n$, since $D_\Sigma|_{U_\sigma}$ is trivial. Hence, $\nu(\chi^m) = \phi(m)$ and thus $(\phi(m),\ell) \in C\cap(\mathbb{Z}^n\times\mathbb{Z})$.

Now, since ϕ is injective and we have precisely $h^0(X_\Sigma, \ell \boldsymbol{D})$ lattice points in $\Delta_{\ell \boldsymbol{D}}$, we have that for all $\ell \geq 0$,

$$C \cap (\mathbb{Z}^n \times \{\ell\}) = \operatorname{Im}\left(v \colon H^0(X_\Sigma, \ell \boldsymbol{D}) \setminus \{0\} \longrightarrow \mathbb{Z}^n\right) = \phi(\Delta_{\ell \boldsymbol{D}} \cap M) = C' \cap (\mathbb{Z}^n \times \{\ell\}),$$

and hence

$$C \cap (\mathbb{Z}^n \times \mathbb{Z}) = C' \cap (\mathbb{Z}^n \times \mathbb{Z}),$$

as we wanted to show.

■

Remark 2.1.16. The assumption in the above theorem that $D_\Sigma|_{U_\sigma}$ is trivial can always be achieved by passing to a linearly equivalent divisor $D_\Sigma + \operatorname{div}(\chi^m)$. The corresponding polytope P_{D_Σ} is then translated accordingly to $P_{D_\Sigma} - m$.

2.2 Applications

Throughout this section, $\Sigma \subseteq N_\mathbb{R}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. Furthermore, $\boldsymbol{D}$ will denote a toric b-divisor on X_Σ and $A_{\boldsymbol{D}} \subseteq F[t]$ its associated algebra of almost integral type. We now give some applications of the identification of b-convex sets with Okounkov bodies. First, we make a statement regarding the growth of the Hilbert function of the graded subalgebra $A_{\boldsymbol{D}}$. Then, in the big and nef case, we construct a *global Okounkov b-body* generalizing the global Okounkov body constructed in [LM09]. This is an interesting tool which one can use in order to deeper study the birational geometry of toric varieties.

Asymptotic growth of Hilbert functions of algebras of almost integral type

Let $H_{A_{\boldsymbol{D}}}$ be the Hilbert function of the subalgebra algebra $A_{\boldsymbol{D}}$. Before describing the asymptotic behavior of $H_{A_{\boldsymbol{D}}}$, we make the following remark.

Remark 2.2.1. The Hilbert function $H_S \colon \mathbb{N} \to \mathbb{N}$, of a semigroup $S \subseteq \mathbb{Z}^n \times \mathbb{Z}$ is given by the assignment

$$t \longmapsto \#\operatorname{cone}(S) \cap (\mathbb{Z}^n \times \{t\}).$$

By [KK12, Proposition 2.27] the function $H_{A_{\boldsymbol{D}}}$ coincides with $H_{S(A_{\boldsymbol{D}})}$. Moreover, recall the Ehrhart function $f_{\boldsymbol{D}} = f_{\Delta_{\boldsymbol{D}}} \colon \mathbb{N} \to \mathbb{N}$ of the convex set $\Delta_{\boldsymbol{D}}$ (Definition 1.5.14). Then the proof of Theorem 2.1.15 implies that

$$f_{\boldsymbol{D}} = H_{S(A_{\boldsymbol{D}})} = H_{A_{\boldsymbol{D}}}.$$

We have the following proposition.

Proposition 2.2.2. *Let $\boldsymbol{D}$ be a toric b-divisor. The Hilbert function $H_{A_{\boldsymbol{D}}}(t)$ of the graded subalgebra $A_{\boldsymbol{D}}$ grows like $a_n t^n$, where the n-th growth coefficient a_n is equal to the volume of $\Delta_{\boldsymbol{D}}$.*

Proof. This follows from [KK12, Corollary 3.11] and Theorem 2.1.15. ■

The global convex b-cone

We construct a *global* Okounkov b-body in the big and nef case, generalizing the global Okounkov body associated to a projective variety together with a big divisor constructed in [LM09].

For any smooth and complete variety X of dimension n, we denote by $N^1(X) \subseteq H^2(X,\mathbb{R})$ the Néron–Severi group of numerical equivalence classes of divisors on X. This is a free abelian group of finite rank. The corresponding finite dimensional $\mathbb{Q}$- and $\mathbb{R}$-vector spaces are denoted by $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$, respectively.

For a pair (X,D) consisting of a smooth n-dimensional projective variety together with a big divisor $D \subseteq X$, the authors in [LM09] construct a convex Okounkov body $\Delta_D^{Y_\bullet} \subseteq \mathbb{R}^n$. The construction depends on the choice of a complete flag $Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{pt\}$ in X. Here it is also shown that $\Delta_D^{Y_\bullet}$ is independent of the choice of the numerical class $[D]$ in $N^1(X)$ (see [LM09, Proposition 4.1]). The construction of the Okounkov body $\Delta_D^{Y_\bullet}$ can be seen as a special case of the construction described in the previous section. Indeed, given a complete flag $Y_\bullet$, one can define a valuation $\nu_{Y_\bullet}$ induced by this flag which is similar to the valuation given in Definition 2.1.14 (see [LM09, Section 1.1]). Then the convex bodies $\Delta_D^{Y_\bullet}$ and Δ_{A_D} coincide (see [KK12, Theorem 3.9]). Moreover, in [LM09] the authors show that these convex bodies fit together in a nice way while $[D]$ runs through the set of numerical equivalence classes. The following is [LM09, Theorem 4.5].

Theorem 2.2.3. *Let X be a smooth, projective variety of dimension n. There exists a closed convex cone*

$$\Delta(X) \subseteq \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$$

characterized by the property that in the diagram

$$\begin{tikzcd} \Delta(X) \arrow[rr, hook] \arrow[dr] & & \mathbb{R}^n \times N^1(X)_{\mathbb{R}} \arrow[dl, "\mathrm{pr}_2"] \\ & N^1(X)_{\mathbb{R}} & \end{tikzcd}$$

the fiber of $\Delta(X) \to N^1(X)_{\mathbb{R}}$ over any big class $[D] \in N^1(X)_{\mathbb{Q}}$ is Δ_{A_D}, i.e. we have that

$$\mathrm{pr}_2^{-1}([D]) \cap \Delta(X) = \Delta_{A_D} \subseteq \mathbb{R}^n \times \{[D]\} = \mathbb{R}^n.$$

The convex cone $\Delta(X)$ is referred to as the global Okounkov body *of X.*

We have the following remark describing the image of $\Delta(X)$ in $N^1(X)_{\mathbb{R}}$.

Remark 2.2.4. The image of $\Delta(X)$ in $N^1(X)_{\mathbb{R}}$ is the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$, i.e. the closure of the cone spanned by all numerical classes of effective divisors, whose interior is the big cone $\mathrm{Big}(X)$ (see the proof of [LM09, Theorem 4.5]).

We now describe the global Okounkov body $\Delta(X_\Sigma)$ of the smooth and complete toric variety X_Σ. For this, we follow mainly [LM09, Section 6.1]. One can show that starting with a choice of a complete flag $Y_\bullet$ consisting of torus invariant subvarieties and the choice of a basis of $M \simeq \mathbb{Z}^n$ inducing the isomorphism $\phi_{\mathbb{R}} : M_{\mathbb{R}} \simeq \mathbb{R}^n$ from (2.3) and assuming that $D|_{U_\sigma}$ is trivial, one has that

$$\Delta_D^{Y_\bullet} = \phi_{\mathbb{R}}(P_D)$$

for any big toric divisor $D \subseteq X_\Sigma$. Here, as usual, P_D denotes the polytope associated to the toric divisor D (see Section 1.1). Furthermore, we have the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^r \xrightarrow{q} \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where $r = \#\Sigma(1)$. Also, in this special toric case, linear equivalence agrees with numerical equivalence, i.e. we have that $N^1(X_\Sigma) = \mathrm{Cl}(X_\Sigma) = \mathrm{Pic}(X_\Sigma)$ has no torsion. The choice of a basis for M gives us the dual basis for $N = M^\vee$. This in turn gives us a splitting of the above exact sequence and consequently, an isomorphism

$$\beta \colon \mathbb{Z}^n \times N^1(X_\Sigma) \simeq \mathbb{Z}^r,$$

such that $\beta^{-1}(D) = (\mathrm{pr}_1(D), q(D))$, where $\mathrm{pr}_1 \colon \mathbb{Z}^r \to \mathbb{Z}^n$ denotes the projection onto the first n components. By [LM09, part 2 of Proposition 6.1] we have the following description of the global Okounkov body $\Delta(X_\Sigma)$ of Theorem 2.2.3 in the toric case: the convex set $\Delta(X_\Sigma)$ is the inverse image under the isomorphism

$$\beta_{\mathbb{R}} \colon \mathbb{R}^n \times N^1(X_\Sigma)_{\mathbb{R}} \simeq \mathbb{R}^r$$

of the non-negative orthant $\mathbb{R}^r_+ \subseteq \mathbb{R}^r$.

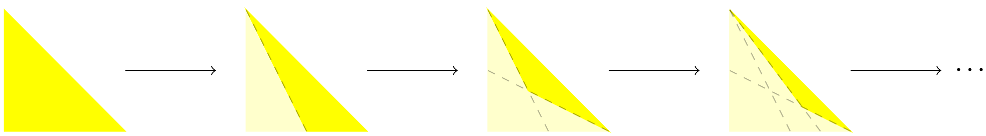
Our goal is to extend Theorem 2.2.3 to the case of toric big and nef b-divisors. Let us start by giving the definition of a *big* toric b-divisor.

Definition 2.2.5. A toric b-divisor $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ is *big* if it has positive volume, i.e. if

$$\mathrm{vol}(\boldsymbol{D}) := \limsup_{\ell \to \infty} \frac{n! h^0(X_\Sigma, \ell \boldsymbol{D})}{\ell^n} > 0.$$

Remark 2.2.6. It follows from the definitions that a toric b-divisor $\boldsymbol{D}$ is big if and only if its corresponding convex set $\Delta_{\boldsymbol{D}}$ is full-dimensional. This generalizes the classical theory of toric divisors, where it is known that bigness of a toric divisor D is equivalent to the full-dimensionality of the polyhedron P_D (see the discussion after [CLS10, Lemma 9.3.0]).

Remark 2.2.7. Let $\boldsymbol{D}$ be a nef toric b-divisor on X_Σ. We can ask whether bigness of $\boldsymbol{D}$ is equivalent to each of the incarnations $D_{\Sigma'}$, for $\Sigma' \in R(\Sigma)$, being big. One implication is clear, since if we start with a big toric b-divisor, then if one of the polytopes $P_{D_{\Sigma'}}$ where not to have full-dimensional volume, then since $\Delta_{\boldsymbol{D}} \subseteq P_{D_{\Sigma'}}$, this would contradict the bigness hypothesis. The other implication however is not true, i.e. we can produce a nef toric b-divisor, all whose incarnations are big, but it itself is not. This follows from the fact that we can produce a sequence of concave virtual support functions compatible under push-forward, with full-dimensional stability sets, converging pointwise to a concave function whose stability set is of strictly smaller dimension. Indeed consider the following sequence of convex sets and the corresponding support functions. This converges to the 1-dimensional segment $[(0,1),(1,0)]$ in $\mathbb{R}^2$.



Definition 2.2.8. We define the space of numerical classes $N^1(\mathfrak{X}_\Sigma)$ of the toric Riemann–Zariski space $\mathfrak{X}_\Sigma$ as the inverse limit

$$N^1(\mathfrak{X}_\Sigma) := \varprojlim_{\Sigma' \in R(\Sigma)} N^1(X_{\Sigma'})$$

with maps given by the push-forward map of numerical classes of Cartier divisors. Moreover, we define the topological space $N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}}$ by

$$N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}} := \varprojlim_{\Sigma' \in R(\Sigma)} N^1(X_{\Sigma'})_{\mathbb{R}}.$$

Remark 2.2.9. Recall that in the case of smooth toric varieties numerical equivalence of divisors agrees with linear equivalence. Hence, the following definition makes sense.

Definition 2.2.10. The Picard group $\operatorname{Pic}(\mathfrak{X}_\Sigma)$ of the toric Riemann–Zariski space $\mathfrak{X}_\Sigma$ is defined as the inverse limit

$$\operatorname{Pic}(\mathfrak{X}_\Sigma) := \varprojlim_{\Sigma' \in R(\Sigma)} \operatorname{Pic}(X_{\Sigma'})$$

with maps given by the push-forward map of linear (= numerical) equivalence classes of Cartier divisors. Moreover, we define the topological space $\operatorname{Pic}(\mathfrak{X}_\Sigma)_{\mathbb{R}}$ by

$$\operatorname{Pic}(\mathfrak{X}_\Sigma)_{\mathbb{R}} := \varprojlim_{\Sigma' \in R(\Sigma)} \operatorname{Pic}(X_{\Sigma'})_{\mathbb{R}}.$$

Remark 2.2.11. (1) Since in the case of smooth toric varieties numerical equivalence of divisors agrees with linear equivalence, we have that this is also true for the toric Riemann–Zariski space, i.e. we have that

$$N^1(\mathfrak{X}_\Sigma) = \operatorname{Pic}(\mathfrak{X}_\Sigma).$$

We will write $[\boldsymbol{D}]$ to denote the class of the toric b-divisor $\boldsymbol{D}$ in $N^1(\mathfrak{X}_\Sigma)$.
(2) Note that $N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}}$ is an inverse limit of finite dimensional vector spaces, but it is itself not finite dimensional. We view it just as a topological vector space (with the topology induced by the inverse limit topology).

The following is the analogue of Theorem 2.2.3 for toric big and nef b-divisors.

Theorem 2.2.12. *There exists a closed convex cone*

$$\Delta(\mathfrak{X}_\Sigma) \subseteq \mathbb{R}^n \times N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}}$$

characterized by the property that in the diagram

$$\begin{tikzcd}
\Delta(\mathfrak{X}_\Sigma) \arrow[rr, hook] \arrow[dr] & & \mathbb{R}^n \times N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}} \arrow[dl, "\mathrm{pr}_2"] \\
& N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}} &
\end{tikzcd}$$

the fiber of $\Delta(\mathfrak{X}_\Sigma) \to N^1(\mathfrak{X}_\Sigma)_{\mathbb{R}}$ over any big and nef class $[\boldsymbol{D}]$ in $N^1(\mathfrak{X}_\Sigma)_{\mathbb{Q}}$ is

$$\phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}}) = \phi_{\mathbb{R}}\left(\Delta_{[\boldsymbol{D}]}\right) = \Delta_{A_{\boldsymbol{D}}},$$

where $\phi_{\mathbb{R}} \colon M_{\mathbb{R}} \simeq \mathbb{R}^n$ is the identification of (2.3), i.e. we have that

$$\mathrm{pr}_2^{-1}([\boldsymbol{D}]) \cap \Delta(\mathfrak{X}_\Sigma) = \phi_{\mathbb{R}}(\Delta_{\boldsymbol{D}}) \subseteq \mathbb{R}^n \times \{[\boldsymbol{D}]\} = \mathbb{R}^n.$$

The convex cone $\Delta(\mathfrak{X}_\Sigma)$ is referred to as the global Okounkov b-body *of X_Σ.*

Proof. We define

$$\Delta(\mathfrak{X}_\Sigma) := \varprojlim_{\Sigma' \in R(\Sigma)} \Delta(X_{\Sigma'}) \subseteq \mathbb{R}^n \times N^1(\mathfrak{X}_\Sigma)_\mathbb{R},$$

where $\Delta(X_{\Sigma'})$ is the global Okounkov body of the smooth and complete toric variety $X_{\Sigma'}$. This is a well defined closed set in the topological space $\mathbb{R}^n \times N^1(\mathfrak{X}_\Sigma)_\mathbb{R}$. Indeed, the topology of the space $\mathbb{R}^n \times N^1(\mathfrak{X}_\Sigma)_\mathbb{R}$ can be seen as a subspace topology of the product topology given on $\prod_{\Sigma' \in R(\Sigma)} \mathbb{R}^n \times N^1(X_{\Sigma'})_\mathbb{R}$. Hence, since all the sets $\Delta(X_{\Sigma'})$ are closed in $\mathbb{R}^n \times N^1(X_{\Sigma'})_\mathbb{R}$, their product and hence their inverse limit is a closed set in $\mathbb{R}^n \times N^1(\mathfrak{X}_\Sigma)_\mathbb{R}$. Moreover, let $[\boldsymbol{D}] \in N^1(\mathfrak{X}_\Sigma)_\mathbb{Q}$ be the class of a big and nef toric b-divisor. Since for all $\Sigma' \in R(\Sigma)$, the Okounkov bodies $\Delta_{A_{D_{\Sigma'}}}$ only depend on the class $[D_{\Sigma'}]$ of the toric divisor and since, by nefness, $\Delta_{A_{\boldsymbol{D}}}$ can be seen as the limit of the $\Delta_{A_{D_{\Sigma'}}}$ as Σ' runs through $R(\Sigma)$, we conclude that $\Delta_{A_{\boldsymbol{D}}}$ also only depends on the class. Whence, we have

$$\begin{aligned}
\mathrm{pr}_2^{-1}(\boldsymbol{D}) \cap \Delta(\mathfrak{X}_\Sigma) &= \mathrm{pr}_2^{-1}\left(\varprojlim_{\Sigma' \in R(\Sigma)} D_{\Sigma'}\right) \cap \varprojlim_{\Sigma' \in R(\Sigma)} \Delta(X_{\Sigma'}) \\
&= \varprojlim_{\Sigma' \in R(\Sigma)} \left(\mathrm{pr}_2^{-1}(D_{\Sigma'}) \cap \Delta(X_{\Sigma'})\right) \\
&= \varprojlim_{\Sigma' \in R(\Sigma)} \Delta_{A_{D_{\Sigma'}}} \times [\boldsymbol{D}] \quad \left(\subseteq \mathbb{R}^n \times \varprojlim_{\Sigma' \in R(\Sigma)} N^1(X_{\Sigma'})_\mathbb{R}\right) \\
&= \Delta_{A_{\boldsymbol{D}}} = \phi_\mathbb{R}(\Delta_{\boldsymbol{D}}),
\end{aligned}$$

where the third equality follows from Theorem 2.2.3 and the last one follows from Theorem 2.1.15. ■

Remark 2.2.13. By Remark 2.2.4 we can describe the image of $\Delta(\mathfrak{X}_\Sigma)$ in $N^1(\mathfrak{X}_\Sigma)_\mathbb{R}$ under the projection pr_2 as follows: for any $\Sigma' \in R(\Sigma)$, the incarnation in the vector space $N^1(X_{\Sigma'})_\mathbb{R}$ of the image of $\Delta(\mathfrak{X}_\Sigma)$ in $N^1(\mathfrak{X}_\Sigma)_\mathbb{R}$ is the pseudoeffective cone $\overline{\mathrm{Eff}(X_{\Sigma'})}$.

We denote by $\mathrm{Big}(\mathfrak{X}_\Sigma)$ and by $\mathrm{Nef}(\mathfrak{X}_\Sigma)$ the set of classes of big and nef toric b-divisors on X_Σ, respectively. The following corollary is the analogue of [KK12, Corollary 4.12] in the b-context.

Corollary 2.2.14. *There is a uniquely defined continuous function*

$$\deg_{\mathfrak{X}_\Sigma} \colon \mathrm{Big}(\mathfrak{X}_\Sigma) \cap \mathrm{Nef}(\mathfrak{X}_\Sigma) \longrightarrow \mathbb{R}$$

that computes the degree of any big and nef toric b-divisor class. This function is homogeneous of degree n and log-concave, i.e

$$\deg_{\mathfrak{X}_\Sigma}(\boldsymbol{D}+\boldsymbol{E})^{1/n} \geq \deg_{\mathfrak{X}_\Sigma}(\boldsymbol{D})^{1/n} + \deg_{\mathfrak{X}_\Sigma}(\boldsymbol{E})^{1/n}$$

for any $\boldsymbol{D}, \boldsymbol{E}$ *in* $\mathrm{Big}(\mathfrak{X}_\Sigma) \cap \mathrm{Nef}(\mathfrak{X}_\Sigma)$.

Proof. We take

$$\deg_{\mathfrak{X}_\Sigma} = \boldsymbol{D}^n = n!\,\mathrm{vol}(\Delta_{\boldsymbol{D}}).$$

Note that $\mathrm{vol}(\Delta_{\boldsymbol{D}}) = \mathrm{vol}(\Delta_{A_{\boldsymbol{D}}}) = \mathrm{vol}(\Delta(\mathfrak{X}_\Sigma)_{\boldsymbol{D}})$ corresponds to the volume of the fibre of the projection pr_2. Then, log-concavity follows from the Brunn–Minkowski inequality in Corollary 1.3.11. And since concave functions are continuous in the interior of their domain, the statement of the corollary follows. ■

Chapter 3

Connection with tropical intersection theory

In this chapter we forget the algebraic side and focus only on combinatorial aspects. We consider a generalization of the concept of a rational polyhedral fan, namely a so called combinatorially principal, weakly embedded conical complex Π (Definitions 3.1.1, 3.1.13 and 3.2.8). We introduce a net of discrete measures supported on the rays of the conical complex, ranging over smooth subdivisions of the complex, with weights defined via tropical intersection theory. One of the goals of this chapter is to prove that under suitable positivity assumptions, the net of discrete measures weakly converges to a limit measure on a compact subset $\mathbb{S}^{\Pi} \subseteq |\Pi|$. We also define a mixed limit measure version of these limit measures. Moreover, we show that if Π is a smooth and complete fan, then this limit measure and the mixed version thereof is related to the surface area measure and to the mixed surface area measure associated to a convex body and to a collection of convex bodies, respectively. For the latter objects we mainly refer to the survey of Schneider ([Sch93]). Assuming some smoothness conditions, this relation enables us to compute integrals with respect to these limit measures explicitly in terms of Lebesgue measures of determinants of Hessians of smooth functions. As an application, we end this chapter by giving a canonical decomposition of the difference $K_2 \setminus K_1$ of two convex sets and by interpreting the volumes of the pieces geometrically as tropical intersection numbers.

The result about the convergence of the discrete measures is the key ingredient which will allow us to prove an integrability result for nef toroidal b-divisors in the following chapter. The connection with the surface area measure and the mixed version theoreof will allow us to write down explicit formulas for computing the degrees and the mixed degrees of toroidal b-divisors, respectively.

The definition of the discrete measures (Definition 3.2.23) seems to be new as well as the method of proof for the weak convergence given in Section 3.3. Also the concept of b-divisors on a conical complex seems to be given here for the first time. As for the canonical decomposition given in Section 3.5, in the polyhedral case, this canonical decomposition gives a polyhedral subdivision of the complement of two polytopes, one contained in the other. This subdivision appears in the literature (e.g. in [GP88]) although it is constructed using the so called pushing method. We haven't found in the literature the method we used in Proposition 3.5.21 nor have we found such a canonical decomposition in the non-polyhedral case.

3.1 Weakly embedded rational conical polyhedral complexes

In this section, we recall the definition of a (weakly embedded) rational conical polyhedral complex. We also define projective subdivisions of such a complex. In the next chapter we will see that on the geometric side, one can naturally attach such a weakly embedded rational conical polyhedral complex to any toroidal embedding. Moreover, projective subdivisions of this complex will correspond to some special toroidal, proper, birational modifications of the toroidal embedding. We refer to [KKMSD73] or [Pay09] and to [Gro15] for definitions and basic results regarding rational conical polyhedral complexes and weak embeddings thereof, respectively.

The following definition is adapted from [Pay09, Definition 2.1].

Definition 3.1.1. A *rational conical polyhedral complex* is a triple

$$\Pi = (|\Pi|, \{\sigma^\alpha\}_{\alpha\in\Lambda}, \{M^\alpha\}_{\alpha\in\Lambda})$$

consisting of a connected topological space $|\Pi|$ together with a finite collection of closed subsets $\sigma^\alpha \subseteq |\Pi|$ and for each σ^α, a finitely generated $\mathbb{Z}$-module M^α of continuous, $\mathbb{R}$-valued functions on σ^α satisfying the following conditions. Let $N^\alpha := \mathrm{Hom}(M^\alpha, \mathbb{Z})$ denote the dual lattice.

(1) For each $\alpha \in \Lambda$, the evaluation map $\phi^\alpha\colon \sigma^\alpha \to N^\alpha_\mathbb{R}$ given by the assignment

$$v \longmapsto (u \mapsto u(v)) \quad (u \in M^\alpha),$$

maps σ^α homeomorphically to a strictly convex, full-dimensional, rational polyhedral cone in $N^\alpha_\mathbb{R}$. We will call the sets σ^α for $\alpha \in \Lambda$ *cones*.

(2) The preimage under ϕ^α of each face of $\phi^\alpha(\sigma^\alpha)$ is $\sigma^{\alpha'}$ for some index $\alpha' \in \Lambda$, and we have that $M^{\alpha'} = \left\{ u|_{\sigma^{\alpha'}} \,\middle|\, u \in M^\alpha \right\}$.

(3) The topological space $|\Pi|$ is the disjoint union of the relative interiors of all the cones σ^α. (By the relative interior of a cone σ^α we mean the preimage under ϕ^α of the relative interior of the rational polyhedral cone $\phi^\alpha(\sigma^\alpha) \subseteq N^\alpha_\mathbb{R}$.)

The $\mathbb{Z}$-modules M^α give the complex a so called *integral structure*. A rational conical polyhedral complex is *smooth* if every cone $\sigma^\alpha \in \Pi$ is smooth, i.e. if $\phi^\alpha(\sigma^\alpha)$ is generated by a $\mathbb{Z}$-basis of $N^\alpha_\mathbb{R}$. The *dimension* of the rational conical polyhedral complex Π is $\max_{\alpha\in\Lambda}\left\{\dim\left(M^\alpha_\mathbb{R}\right)\right\}$.

Given a cone $\sigma \in \Pi$, we will write M^σ, N^σ and ϕ^σ for the corresponding lattice, dual lattice and evaluation map, respectively.

Notation. We will usually refer to a *rational conical polyhedral complex* just as a *conical complex*. We also denote by $\langle , \rangle_\alpha$ the pairing induced by the dual lattices M^α and N^α. We will usually omit the index "α" from the pairing. Finally, for $d \geq 0$, we write $\Pi(d)$ for the set of d-dimensional cones in a conical complex Π.

The following remark follows from [Pay09, Remark 2.6].

Remark 3.1.2. Connectedness of Π implies that Π has a unique minimal cone, i.e. a cone which has no proper faces. Then, since we made the requirement that the cones in the conical complex are strictly convex, we conclude that a conical complex Π contains a unique 0-dimensional cone $\{0_\Pi\}$.

Example 3.1.3. Let N be a lattice, $M = N^\vee$ its dual lattice and let $\Sigma \subseteq N_\mathbb{R}$ be a rational polyhedral fan. Then the triple $\left(|\Sigma|, \{\sigma\}_{\sigma\in\Sigma}, \{M^\sigma\}_{\sigma\in\Sigma}\right)$, where M^σ is the quotient lattice $M/\left(\sigma^\perp \cap M\right)$, is an example of a conical complex. Here, $\sigma^\perp$ denotes the set $\{m \in M \,|\, \langle m, v\rangle = 0, \forall v \in \sigma\}$.

Remark 3.1.4. Note that the topological space $|\Pi|$ of a conical complex does not necessarily come equipped with an embedding into a vector space and in particular the lattices $\{M^\alpha\}$ do not necessarily arise as the quotients of a single fixed lattice as in the case of a fan $\Sigma \subseteq N_\mathbb{R}$.

Definition 3.1.5. A *morphism* $f\colon \Theta \to \Pi$ of conical complexes is a continuous map $|\Theta| \to |\Pi|$ with the property that for every cone $\tau \in \Theta$ there exists a cone $\sigma \in \Pi$ such that $f(\tau) \subseteq \sigma$, and that the restriction $f|_\tau\colon \tau \to \sigma$ is a morphism of integral cones, i.e. the restriction $f|_\tau$ is continuous and satisfies that $m \circ f|_\tau$ is in M^τ for all m in M^σ.

The following definition is taken from [KKMSD73, Definition 2, pg. 86].

Definition 3.1.6. Let $\Pi = (|\Pi|, \{\sigma^\alpha\}_{\alpha\in\Lambda}, \{M^\alpha\}_{\alpha\in\Lambda})$ be a conical complex. A *finite partial polyhedral decomposition* is a second conical complex $\Pi' = \left(|\Pi'|, \{\sigma^{\alpha'}\}_{\alpha'\in\Lambda'}, \{M^{\alpha'}\}_{\alpha'\in\Lambda'}\right)$ such that the following is satisfied:

(1) $|\Pi'| \subseteq |\Pi|$,

(2) for all indices $\alpha' \in \Lambda'$ there exists an index $\alpha \in \Lambda$ such that $\operatorname{relint}\left(\sigma^{\alpha'}\right) \subseteq \operatorname{relint}(\sigma^\alpha)$,

(3) if $\sigma^{\alpha'} \subseteq \sigma^\alpha$ for some indices $\alpha' \in \Lambda'$ and $\alpha \in \Lambda$, then $M^{\alpha'} = M^\alpha|_{\sigma^{\alpha'}}$.

We will refer to a finite partial polyhedral decomposition just as a *decomposition*. Decompositions having $|\Pi'| = |\Pi|$ are called *subdivisions*.

Definition 3.1.7. Let Π be a conical complex. We define the directed set $R(\Pi)$ to be the set of all smooth subdivisions of Π with the partial order given by

$$\Pi'' \geq \Pi' \text{ in } R(\Pi) \quad \text{iff} \quad \Pi'' \text{ is a smooth subdivision of } \Pi'.$$

The following definition ([KKMSD73, Definition 1.5, pg. 111]) can be seen as the analogue of a projective fan, i.e. a fan which is the normal fan of a rational polytope.

Definition 3.1.8. Let Π be a conical complex. A subdivision $\Pi' \in R(\Pi)$ is said to be *projective* if there exists a function $\phi\colon |\Pi| \to \mathbb{R}$ satisfying the following properties:

(1) ϕ is conical, i.e. $\phi(\lambda \cdot v) = \lambda \cdot \phi(v)$ for all $\lambda \in \mathbb{R}_{\geq 0}$ and $v \in |\Pi|$.

(2) ϕ is continuous and piecewise linear.

(3) Π' is determined by the regions of linearity of ϕ, i.e. the cones of Π' correspond to the regions of linearity of the function ϕ.

(4) ϕ is rational, i.e. $\phi|_{\sigma_{\alpha'}}$ belongs to $M^{\alpha'}_\mathbb{Q}$ for all $\alpha' \in \Lambda'$.

(5) ϕ is concave on each cone σ^α.

Remark 3.1.9. It follows from [KKMSD73, Corollary 1.12] that projective subdivisions are transitive, i.e. if $\Pi'' \geq \Pi'$ and $\Pi' \geq \Pi$ are projective subdivisions, then $\Pi'' \geq \Pi$ is a projective subdivision.

The next example appears in [KKMSD73, Example 2.1].

Example 3.1.10. Let $\Pi := \Pi^{-1}$ be an n-dimensional smooth conical complex. For $0 \leq k \leq n-2$ we let Π^k be the subdivision of Π^{k-1} obtained by taking the barycentric subdivision of each of the $(n-k)$-dimensional cones σ in Π (see Definitions 1.2.9 and 1.2.10 for the definition of the barycentric subdivision of a cone). We then have a succession of subdivisions

$$\Pi := \Pi^{-1} \leq \Pi^0 \leq \Pi^1 \leq \cdots \leq \Pi^{n-2} = \Pi'.$$

Π' is called the *barycentric subdivision of* Π. Now, let $f_k\colon \left|\Pi^{k-1}\right| \to \mathbb{R}$ be the function which has value 1 on all the barycenters of the $(n-k)$-dimensional cones in Π, 0 on all the rays of Π^{k-1} and is linear on each cone in Π^k. Then f_k satisfies all the properties of Definition 3.1.8 with respect to Π^{k-1}, i.e. f_k turns Π^k into a projective subdivision of Π^{k-1}. Thus, by the transitivity of projective subdivisions, we get that the barycentric subdivision Π' of Π is projective. Moreover, if Π is smooth then Π' is smooth as well. Figure 3.1 gives an example of a barycentric subdivision in dimension 3.

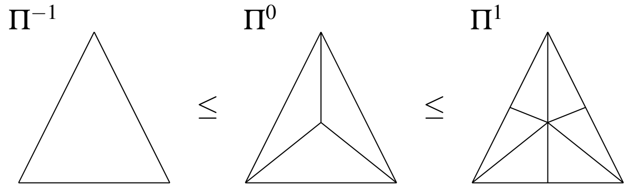


Figure 3.1: 3-dimensional fan cut by the plane $(z = 1)$

We make the following definition.

Definition 3.1.11. Let Π be a conical complex. We define the *directed* set $R'(\Pi) \subseteq R(\Pi)$ to be the set of all smooth projective subdivisions of Π with the induced partial order.

Lemma 3.1.12. *The subset $R'(\Pi) \subseteq R(\Pi)$ is cofinal.*

Proof. Barycentric subdivisions of Π belong to $R'(\Pi)$ and any smooth subdivision is dominated by a barycentric one (see Remark 1.2.12). ■

As was mentioned at the beginning of this chapter, later on, we will compute tropical intersections on a conical complex. For this, we need the so called *balancing condition* to be satisfied at any cone in the complex (see Definition 3.2.2). However, when gluing two cones σ and σ' along a common face, there is no natural lattice containing both N^σ and $N^{\sigma'}$ (see Remark 3.1.4). Hence, a priori, we cannot talk about balancing. To fix this, one makes the following definition ([Gro15, Definition 2.1]) which interpolates between polyhedral fans and polyhedral conical complexes.

Definition 3.1.13. A *weakly embedded conical complex* is a conical complex Π, together with a lattice N^Π, and a continuous map $\iota_\Pi\colon |\Pi| \to N^\Pi_\mathbb{R}$ which is integral linear on every cone σ of Π, i.e. such that for every cone $\sigma \in \Pi$, we have that $\iota_\Pi|_{|\sigma|} = \varphi^\sigma \circ \phi^\sigma$ for some integral linear function $\varphi^\sigma\colon N^\sigma_\mathbb{R} \to N^\Pi_\mathbb{R}$.

Remark 3.1.14. The notion of a fan is recovered by imposing the additional requirements that ι_Π is injective, and that the lattices spanned by $\iota_\Pi(N^\sigma \cap \sigma)$ for $\sigma \in \Pi$ are saturated in N^Π. On the other hand, the notion of a conical complex is recovered by setting $N^\Pi = 0$.

Given a weakly embedded conical complex Π we write $M^\Pi = \operatorname{Hom}\left(N^\Pi, \mathbb{Z}\right)$ for the dual lattice of N^Π. For every cone $\sigma \in \Pi$ we write N^Π_σ for $N^\Pi \cap \operatorname{Span}(\iota_\Pi(\sigma))$ and $M^\Pi_\sigma = \operatorname{Hom}\left(N^\Pi_\sigma, \mathbb{Z}\right)$ for its dual.

Definition 3.1.15. A *morphism between two weakly embedded conical complexes* Θ and Π consists of a morphism of conical complexes $\Theta \to \Pi$ together with a morphism of lattices $N^{\Theta} \to N^{\Pi}$ forming a commutative square with the weak embeddings.

We come to the analogue of the concept of the star of a fan at a cone (see Section 1.1).

Definition 3.1.16. Let Π be a conical complex and let τ be a cone in Π. For every cone σ in Π containing τ let σ/τ be the image of σ in $N^{\sigma}_{\mathbb{R}}/N^{\tau}_{\mathbb{R}}$. Whenever σ and σ' are two cones containing τ such that σ' is a face of σ, the cone σ'/τ is naturally identified with a face of σ/τ. Gluing the cones σ/τ for $\tau \leq \sigma \in \Pi$ along these faces produces a new conical complex which is called the *star complex of* Π *at* τ, denoted by $\Pi(\tau)$.

Remark 3.1.17. If Π is a weakly embedded conical complex, then the star is naturally a weakly embedded conical complex again. Indeed, for every cone σ of Π containing τ there is an induced integral linear map $\sigma/\tau \to \left(N^{\Pi}/N^{\Pi}_{\tau}\right)_{\mathbb{R}} =: N^{\Pi(\tau)}_{\mathbb{R}}$. These maps glue to give a continuous map

$$\iota_{\Pi(\tau)} \colon |\Pi(\tau)| \longrightarrow N^{\Pi(\tau)}_{\mathbb{R}}.$$

3.2 The tropical intersection product

Throughout this section Π will denote a smooth, weakly embedded, n-dimensional conical complex with weak embedding given by $\iota_{\Pi} \colon |\Pi| \to N^{\Pi}_{\mathbb{R}}$. The goal of this section is to define the tropical intersection product between tropical cycles (Definition 3.2.3) and combinatorially principal $\mathbb{Q}$-Cartier divisors on Π (Definitions 3.2.4 and 3.2.6). In order to do this we recall the definition of a lattice normal vector and of a Minkowski weight. We end this section by defining the discrete measure μ_{ϕ} associated to a cp $\mathbb{Q}$-Cartier divisor ϕ on Π using the tropical intersection product. Our main reference for this section is [Gro15]. For the interested reader, the articles [AR10], [FS97] and [Kat12] constitute a more thorough reference for tropical intersection theory.

Tropical cycles and Cartier divisors

We start with some definitions. These are adapted from [Gro15, Section 3.1].

Definition 3.2.1. Let τ be a 1-codimensional face of a cone σ in Π. We define the *lattice normal vector* $v_{\sigma/\tau}$ *of* σ *relative to* τ to be the image under the weak embedding of the unique primitive vector in a ray of σ which is not in τ.

By smoothness, this is well defined. Note that lattice normal vectors can be zero.

Definition 3.2.2. A *k-dimensional Minkowski weight* on Π is a map

$$c \colon \Pi(k) \longrightarrow \mathbb{Q},$$

from the set of k-dimensional cones in Π to the rational numbers such that it satisfies the *balancing condition* around every $(k-1)$-dimensional cone $\tau \in \Pi$, i.e. if $\sigma_1, \ldots, \sigma_{\ell}$ are the k-dimensional cones containing τ, then the relation

$$\sum_{i=1}^{\ell} c\left(\sigma_i\right) v_{\sigma_i/\tau} \in N^{\Pi}_{\tau}$$

holds true. The k-dimensional Minkowski weights naturally form an abelian group, which is denoted by $M_k(\Pi)$.

For every subdivision $\Pi' \in R(\Pi)$ of weakly embedded conical complexes there is an induced morphism $g\colon M_k(\Pi) \to M_k(\Pi')$ between the groups of Minkowski weights. Indeed, let $c \in M_k(\Pi)$ and let $f\colon \Pi' \to \Pi$ be the morphism of weakly embedded conical complexes induced by the subdivision. For each $\sigma' \in \Pi'$ let $\sigma \in \Pi$ be the smallest cone containing $f(\sigma')$. Then if σ' is of dimension k, the weight $g(c)$ of σ' is given by

$$g(c)(\sigma') = \begin{cases} c(\sigma) & \text{if } \dim\sigma = \dim\sigma', \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

This is a well defined Minkowski weight (see [Gro15, Remark 3.2]). Hence, the following definition makes sense.

Definition 3.2.3. The group of *tropical k-cycles* on Π is defined as the direct limit

$$Z_k(\Pi) := \varinjlim_{\Pi' \in R(\Pi)} M_k\left(\Pi'\right),$$

with maps given as in (3.1). If c is a k-dimensional Minkowski weight on a subdivision Π' of Π, we denote by $[c]$ its image in $Z_k(\Pi)$.

We now define $\mathbb{Q}$-Cartier divisors on the conical complex Π. The following two definitions are taken from [Gro15, Definition 3.6].

Definition 3.2.4. A *Cartier divisor on* Π is a function $\phi\colon |\Pi| \to \mathbb{R}$ satisfying the following conditions:

(1) ϕ is conical.

(2) ϕ is continuous.

(3) ϕ is linear on each cone $\sigma \in \Pi$.

(4) ϕ is integral, i.e. $\phi|_\sigma$ belongs to M^σ for every cone $\sigma \in \Pi$.

We denote the group of Cartier divisors on Π by $\mathrm{Div}(\Pi)$. The group of $\mathbb{Q}$-Cartier divisors on Π is $\mathrm{Div}(\Pi)_{\mathbb{Q}} := \mathrm{Div}(\Pi) \otimes \mathbb{Q}$.

Remark 3.2.5. Note the similarities of the above definition with Definition 3.1.8. A $\mathbb{Q}$-Cartier divisor on Π is a function satisfying the properties in Definition 3.1.8 but instead of asking for concavity on each cone, we ask for the stronger condition of linearity.

Definition 3.2.6. A Cartier divisor or a $\mathbb{Q}$-Cartier divisor ϕ is said to be *combinatorially principal* (abbreviated cp) if on each cone σ, the linear function $\phi|_\sigma$ is of the form $\langle m_\sigma, \cdot\rangle$ for an element m_σ in M^Π or in $M^\Pi_{\mathbb{Q}}$, respectively. The subgroup of cp Cartier divisors and of cp $\mathbb{Q}$-Cartier divisors on Π are denoted by $\mathrm{cpDiv}(\Pi)$ and by $\mathrm{cpDiv}(\Pi)_{\mathbb{Q}}$, respectively.

Example 3.2.7. If $N^\Pi = 0$ then there are no cp Cartier nor $\mathbb{Q}$-Cartier divisors on Π different from 0. On the other hand, if $\Pi = \Sigma \subseteq N_{\mathbb{R}}$ is a fan for some lattice N and $N^\Sigma = N$, then any Cartier or $\mathbb{Q}$-Cartier divisor on Σ is cp.

The following definition seems to be new.

Definition 3.2.8. The conical complex Π is called *combinatorially principal* (abbreviated cp) if the restriction

$$\iota_\Pi|_\sigma \colon |\sigma| \longrightarrow N_\mathbb{R}^\Pi$$

of the weak embedding to every cone $\sigma \in \Pi$ is injective.

For every cone $\sigma \in \Pi$, let $\varphi^\sigma \colon N_\mathbb{R}^\sigma \to N_\mathbb{R}^\Pi$ be the integral linear function of Definition 3.1.13 such that $\iota_\Pi|_{|\sigma|} = \varphi^\sigma \circ \phi^\sigma$. We have the following lemma.

Lemma 3.2.9. *The conical complex* Π *is cp if and only if every* $\mathbb{Q}$*-Cartier divisor on* Π *is cp.*

Proof. Recall that Π is smooth. It is cp if and only if for every cone $\sigma \in \Pi$, the restriction

$$\varphi^\sigma \colon N_\mathbb{R}^\sigma \longrightarrow N_\mathbb{R}^\Pi$$

is injective. By integrality, this is equivalent to the restriction

$$\varphi^\sigma \colon N_\mathbb{Q}^\sigma \longrightarrow N_\mathbb{Q}^\Pi$$

being injective. And this happens if and only if the dual map

$$M_\mathbb{Q}^\Pi \longrightarrow M_\mathbb{Q}^\sigma$$

is surjective for every cone $\sigma \in \Pi$. Now, if $M_\mathbb{Q}^\Pi \to M_\mathbb{Q}^\sigma$ is surjective for every cone σ in Π, then, clearly, any $\mathbb{Q}$-Cartier divisor on Π is cp. Conversely, assume that any $\mathbb{Q}$-Cartier divisor on Π is cp and let $m \in M_\mathbb{Q}^\sigma$. By smoothness, there exists a $\mathbb{Q}$-Cartier divisor D on Π which is of the form $\langle m, \cdot \rangle$ on σ. Since D is cp, there exists an $m' \in M_\mathbb{Q}^\Pi$ mapping to m via $M_\mathbb{Q}^\Pi \to M_\mathbb{Q}^\sigma$, as we wanted to show. ■

Remark 3.2.10. If we consider Cartier divisors on Π instead of $\mathbb{Q}$-Cartier divisors, then the above lemma is no longer true. For it to be true, we must ask for the additional requirement that the lattice N^σ is saturated in N^Π.

Definition 3.2.11. Two cp Cartier divisors on Π are called *linearly equivalent* if their difference is induced by an element in M^Π. The group of cp Cartier and of cp $\mathbb{Q}$-Cartier divisors modulo linear equivalence are denoted by $\mathrm{cpCl}(\Pi)$ and by $\mathrm{cpCl}(\Pi)_\mathbb{Q}$, respectively.

The tropical intersection product

We construct an intersection product

$$\phi \cdot [c] \in Z_{k-1}(\Pi)$$

between a cp $\mathbb{Q}$-Cartier divisor ϕ on Π and a tropical cycle $[c]$ in $Z_k(\Pi)$. For each $\sigma \in \Pi$ we denote by ϕ_σ a linear function on $N_{\sigma\,\mathbb{R}}^\Pi$ which equals ϕ on σ. This can be done because ϕ is cp. Note that this is not the same as the restriction $\phi|_\sigma$ since the latter is a function on $N_\mathbb{R}^\sigma$ and not on $N_{\sigma\,\mathbb{R}}^\Pi$. If $c \in M_k(\Pi)$ is a Minkowski weight, then the map

$$\phi \cdot c \colon \Pi(k-1) \longrightarrow \mathbb{Q}$$

is defined in the following way: for $\tau \in \Pi(k-1)$, the weight $\phi \cdot c$ of τ is given by the formula

$$(\phi \cdot c)(\tau) := \sum_{\substack{\sigma \in \Pi(k) \\ \sigma \geq \tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right) c(\sigma) + \phi_\tau \left(\sum_{\substack{\sigma \in \Pi(k) \\ \sigma \geq \tau}} c(\sigma) v_{\sigma/\tau} \right).$$

Note that this is well defined since $c \in M_k(\Pi)$ is a k-dimensional Minkowski weight and hence

$$\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau} \in N_\tau^\Pi.$$

Moreover, one checks that $\phi \cdot c$ satisfies the balancing condition and is hence a Minkowski weight in $M_{k-1}(\Pi)$ (see [AR10, Proposition 3.7a]). In general, $[c]$ is represented on a subdivision Π' of Π. Note that we have a natural inclusion morphism $\iota\colon M_{k-1}(\Pi') \to Z_{k-1}(\Pi)$. The intersection product $\phi \cdot c \in Z_{k-1}(\Pi)$ is then defined by

$$[\phi \cdot c] := \iota\left(\phi \cdot c\right).$$

Definition 3.2.12. Let ϕ be a cp $\mathbb{Q}$-Cartier divisor on Π and let $[c] \in Z_k(\Pi)$ be a tropical cycle. We define the intersection product $\phi \cdot [c] \in Z_{k-1}(\Pi)$ by

$$\phi \cdot [c] := [\phi \cdot c],$$

where c is any representative Minkowski weight and the product $[\phi \cdot c]$ is the one described above. This is well defined by [Gro15, Construction 3.12].

One can show that this intersection product factors through linear equivalence. The following lemma follows from [Gro15, Construction 3.14].

Lemma 3.2.13. *For* $1 \leq k \leq n$ $(= \dim(\Pi))$ *there exists a bilinear pairing*

$$\mathrm{cpCl}(\Pi)_{\mathbb{Q}} \times Z_k(\Pi) \longrightarrow Z_{k-1}(\Pi)$$

given by the intersection product given in Definition 3.2.12.

We now give the definition of a balanced conical complex.

Definition 3.2.14. The conical complex Π is called *balanced* if the map

$$[\Pi]\colon \Pi(n) \longrightarrow \mathbb{Q}$$

defined by taking the value 1 on all n-dimensional cones of Π is a Minkowski weight. In this case we denote also by $[\Pi]$ the induced tropical cycle.

Remark 3.2.15. Note that if Π is balanced, then any subdivision $\Pi' \geq \Pi$ in $R(\Pi)$ is balanced as well. Indeed, $[\Pi']$ is the image of $[\Pi]$ under the map $M_n(\Pi) \to M_n(\Pi')$ given in Equation (3.1).

We are now ready to define tropical top intersection numbers of cp $\mathbb{Q}$-Cartier divisors on Π.

Definition 3.2.16. Assume that Π is balanced. Let $\phi_1, \ldots, \phi_n$ be cp $\mathbb{Q}$-Cartier divisors on Π. The tropical top intersection number $\phi_1 \cdots \phi_n$ is defined inductively by

$$\phi_1 \cdots \phi_n := \phi_1 \cdots \phi_n \cdot [\Pi] := \phi_1 \cdots \phi_{n-1} \cdot \left(\phi_n \cdot [\Pi]\right).$$

By Lemma 3.2.13, this induces a multilinear symmetric map

$$\underbrace{\mathrm{cpCl}(\Pi)_{\mathbb{Q}} \times \cdots \times \mathrm{cpCl}(\Pi)_{\mathbb{Q}}}_{n\text{-times}} \longrightarrow \mathbb{Q}.$$

Now, recall the notion of the star $\Pi(\tau)$ of the conical complex Π at a cone τ from Definition 3.1.16. As explained in [Gro15, Construction 3.5], we can restrict a Minkowski weight c in $M_k(\Pi)$ to a Minkowski weight $c|_{\Pi(\tau)}$ in $M_{k-\dim\tau}(\Pi(\tau))$ by assigning $c(\sigma)$ to σ/τ for all $\sigma \in \Pi(k)$ with $\tau \leq \sigma$. If $k < \dim\tau$ we set $c|_{\Pi(\tau)} = 0$.

Moreover, let ϕ be a cp $\mathbb{Q}$-Cartier divisor on Π. Choose any affine function ψ on $|\Pi|$ with the property that $\phi|_\tau = \psi|_\tau$. Then the difference $\phi - \psi$ induces a function on $\Pi(\tau)$ which is rational linear on each face. We denote this function by $\phi(\tau)$. Note that $\phi(\tau)$ is unique up to adding a linear function. The following lemma asserts that one can compute tropical intersection numbers locally on the stars. It will be useful to us later on.

Lemma 3.2.17. *Let $0 \leq k \leq n$. With notations as above, the following holds for all Minkowski weights $c \in M_k(\Pi)$.*

$$\phi(\tau) \cdot c|_{\Pi(\tau)} = (\phi \cdot c)|_{\Pi(\tau)}.$$

Proof. This follows from [Rau15][Proposition 1.1] . ■

Remark 3.2.18. In the same way one can define a tropical intersection product between a cp $\mathbb{R}$-Cartier divisor and an $\mathbb{R}$-valued Minkowski weight.

A projection formula

We give a projection formula with respect to the tropical intersection product which is stated in [Gro15, Proposition 3.16]. Let $f\colon \Theta \to \Pi$ be a morphism of weakly embedded conical complexes. We define the pull-back of a $\mathbb{Q}$-Cartier divisor and the push-forward of a tropical cycle along the morphism f (see [Gro15, Constructions 3.8 and 3.10]).

Definition 3.2.19. Let ϕ be a $\mathbb{Q}$-Cartier divisor on Π. Then the function $f^*\phi$ on $|\Theta|$ defined by

$$f^*\phi := \phi \circ f$$

is a divisor on Θ. It is called the *pull-back of ϕ along f*. Moreover, if ϕ is cp so is $f^*\phi$. The map

$$f^*\colon \mathrm{cpDiv}(\Pi) \longrightarrow \mathrm{cpDiv}(\Theta)$$

is a morphism of abelian groups which factors through linear equivalence. Hence we get a morphism

$$f^*\colon \mathrm{cpCl}(\Pi) \longrightarrow \mathrm{cpCl}(\Theta).$$

Now, let $[c] \in Z_k(\Theta)$ be a tropical cycle. In order to define the push-forward of $[c]$ along f we assume that $[c]$ is represented by a Minkowski weight $c \in M_k(\Theta)$. One can show that there exists a refinement $\Pi' \geq \Pi$ in $R(\Pi)$ such that the image $f(\tau)$, for every k-dimensional cone $\tau \in \Theta(k)$, is a union of cones in Π' (see [Gro15, Construction 3.10]).

Definition 3.2.20. With notations as above, the *push-forward of $[c]$ along f* is the tropical cycle $f_*[c] \in Z_k(\Pi)$ represented by the Minkowski weight

$$f_*c\colon \Pi'(k) \longrightarrow \mathbb{Q},$$

which assigns to any k-dimensional cone σ in Π' the value

$$f_*c(\sigma) := \sum_{\substack{\tau\in\Theta(k)\\ f(\tau)\supseteq\sigma}} [N^\sigma : f(N^\tau)]\, c(\tau),$$

where the sum is over all k-dimensional cones of Θ whose image contains σ. The arguments given in [Gro15, Construction 3.10] show that this is well defined.

The following is a projection formula for the tropical intersection product.

Proposition 3.2.21. *Let $f\colon \Theta \to \Pi$ be a morphism of weakly embedded conical complexes. Let ϕ be a cp $\mathbb{Q}$-Cartier divisor on Π and let $[c] \in Z_k(\Theta)$ be a tropical cycle. Then the projection formula*

$$f_*(f^*\phi \cdot [c]) = \phi \cdot f_*[c]$$

is satisfied.

Proof. This is a special case of [Gro15, Proposition 3.16]. ■

The tropical discrete measures

As was mentioned at the beginning of this chapter, one of the goals is to prove that a certain net of discrete measures, defined via tropical intersection theory on Π, weakly converges. Clearly, these tropical intersection numbers depend on the integral structure of the conical complex. It turns out that to prove this convergence it makes sense to get rid of the integral structure and to introduce a metric structure on Π in terms of which all of the necessary properties of the tropical intersection product can be expressed.

Assume that Π is cp and let $|\cdot|$ be a norm on $N_{\mathbb{R}}^{\Pi}$.

Definition 3.2.22. Let v be an element in $|\Pi|$. We denote by $\hat{v}$ the normalization of v defined by $\hat{v} := v/|\iota_\Pi(v)|$. The compact set $\mathbb{S}^\Pi \subseteq |\Pi|$ is defined by

$$\mathbb{S}^\Pi := \left\{ v \in |\Pi| \,\middle|\, |\iota_\Pi(v)| = 1 \right\}.$$

It is compact since it is the inverse image of a compact space under the weak embedding. Note that the normalization $\hat{v}$ is an element in the compact topological space $\mathbb{S}^\Pi$.

We end this section by defining the discrete measures μ_ϕ for every cp $\mathbb{Q}$-Cartier divisor ϕ on Π.

Definition 3.2.23. Assume in addition that Π is balanced. Then for every $\mathbb{Q}$-Cartier divisor ϕ on Π we define the discrete measure μ_ϕ by

$$\mu_\phi := \sum_{\tau \in \Pi(1)} \left(\phi^{n-1} \cdot [\Pi]\right)(\tau) \cdot |\iota_\Pi(v_\tau)| \cdot \delta_{\hat{v}_\tau},$$

where $\delta_{\hat{v}_\tau}$ denotes the Dirac delta measure supported on $\hat{v}_\tau \in \mathbb{S}^\Pi$ for the primitive vector $v_\tau \in |\Pi|$. We will usually write $|v_\tau|$ although we mean $|\iota_\Pi(v_\tau)|$.

Remark 3.2.24. If $\Pi \subseteq N_{\mathbb{R}}\ (\simeq \mathbb{R}^r)$ is a fan, then we may take the standard euclidean norm $|\cdot|$ on $\mathbb{R}^r$ and view the discrete measure μ_ϕ on the sphere $\mathbb{S}^{r-1}$. This depends on the choice of a basis of $\mathbb{R}^r$.

3.3 Weak convergence of tropical discrete measures

Throughout this section Π will denote a balanced, cp, weakly embedded conical complex of dimension n. We assume that the topology on $N_{\mathbb{R}}^{\Pi}$ is induced by a norm $|\cdot|$ coming from an inner product $\langle , \rangle_{N^\Pi}$. The goal of this section is to prove that given a *continuous tropically nef b*-divisor $\phi = (\phi_{\Pi'})_{\Pi' \in R(\Pi)}$ (Definition 3.3.4), the associated net of discrete measures $\mu_{\phi_{\Pi'}}$ from Definition 3.2.23 weakly converges to a limit measure μ_ϕ on $\mathbb{S}^\Pi$. This is done by introducing the notion of the size of a Minkowski weight, a concept which is independent of the integral structure of Π, but depends on the metric structure given by $|\cdot|$. This allows us to prove some monotonicity properties from which we conclude the weak convergence of the discrete measures.

We start by defining some positivity notions for tropical cycles. We remark that Definition 3.3.1 appears in the literature (see e.g. [Rau15, Section 1.5]), whereas Definition 3.3.2 appears to be new.

Definition 3.3.1. A tropical cycle is called *positive* if it is represented by a positive Minkowski weight, i.e. a Minkowski weight whose weights are non-negative. The sub-semigroups of positive Minkowski weights and of positive tropical cycles of dimension k are denoted by $M_k^+(\Pi)$ and by $Z_k^+(\Pi)$, respectively.

Definition 3.3.2. Let $\mathcal{C}$ be a collection of $\mathbb{Q}$-Cartier divisors defined on subdivisions of Π in $R(\Pi)$. We say that $\mathcal{C}$ is *tropically nef* on Π if the following properties are satisfied:

(1) If $\phi_1,\dots,\phi_r \in \mathcal{C}$ are $\mathbb{Q}$-Cartier divisors on $\Pi_1,\dots,\Pi_r$, respectively, where $\Pi_i \in R(\Pi)$ for all $i=1,\dots,r$, then for some common subdivision $\tilde{\Pi}$ in $R(\Pi)$, the tropical cycle $\phi_1^{\ell_1}\cdots\phi_r^{\ell_r}\cdot[\tilde{\Pi}]$ is a positive tropical cycle in $Z_{n-\sum_{i=1}^r \ell_i}(\Pi)$ for every choice of non-negative integers $\ell_1,\dots,\ell_r$ such that $\sum_{i=1}^r \ell_i \leq n$.

(2) If ϕ and ψ are elements in $\mathcal{C}$ then for every choice of non-negative rational numbers λ and μ, the linear combination $\lambda\phi+\mu\psi$ belongs to $\mathcal{C}$.

Example 3.3.3. We give some examples of collections $\mathcal{C}$ of $\mathbb{Q}$-Cartier divisors on Π which are tropically nef.

(1) The trivial class consisting of the $\mathbb{Q}$-Cartier divisors defined on subdivisions of Π in $R(\Pi)$ which are equal to 0 is an example of a tropically nef collection $\mathcal{C}$ on Π.

(2) The collection consisting of $\mathbb{Q}$-Cartier divisors ϕ defined on subdivisions of Π in $R(\Pi)$ which are restrictions of concave functions on $N_{\mathbb{R}}^{\Pi}$ form a tropically nef collection $\mathcal{C}$ on Π.

(3) The collection consisting of $\mathbb{Q}$-Cartier divisors ϕ defined on subdivisions of Π in $R(\Pi)$ with the property that $\phi\cdot[c]$ is a positive tropical cycle for every positive cycle $[c]\in Z_*^+(\Pi)$ is tropially nef on Π.

We now define b-divisors on the conical complex Π. For this, consider a subdivision $\Pi'\in R(\Pi)$ and let $\phi_{\Pi'}$ be a $\mathbb{Q}$-Cartier divisor on Π'. Then for every conical complex $\tilde{\Pi}\in R(\Pi)$ such that $\tilde{\Pi}\leq\Pi'$, if we let $\pi\colon \Pi'\to\tilde{\Pi}$ be the induced morphism of weakly embedded conical complexes, we denote by $\pi_*(\phi_{\Pi'})$ the $\mathbb{Q}$-Cartier divisor on $\tilde{\Pi}$ defined by the values of $\phi_{\Pi'}$ on the rays of $\tilde{\Pi}$.

Definition 3.3.4. A *b-divisor* on Π is a net of $\mathbb{Q}$-Cartier divisors $(\phi_{\Pi'})_{\Pi'\in R(\Pi)}$, indexed over all smooth subdivisions of Π, which is compatible under the push-forward morphism defined above. A b-divisor on Π is called *continuous* if its restriction to $\mathbb{S}^{\Pi}$ converges uniformly to a limit continuous function ϕ. A continuous, conical function on $\mathbb{S}^{\Pi}$ arising as a limit in this way from a continuous b-divisor $(\phi_{\Pi'})_{\Pi'\in R(\Pi)}$, whose constituents belong to a tropically nef collection $\mathcal{C}$ on Π (in the sense of Definition 3.3.2) is called a *continuous tropically nef b-divisor* on Π.

We make the following remarks.

Remark 3.3.5. (1) Any $\mathbb{Q}$-Cartier divisor $\phi_{\Pi'}$ belonging to some tropically nef collection $\mathcal{C}$ on Π, defined on Π' for some $\Pi'\in R(\Pi)$, can be seen as a continuous tropically nef b-divisor on Π simply by taking $\phi_{\Pi''}=\phi_{\Pi'}$ for every $\Pi''\geq\Pi'$ in $R(\Pi)$.

(2) If $\Pi = \Sigma$ is a fan, then any conical, concave function on $|\Sigma|$ is a continuous tropically nef b-divisor on Σ.
(3) We will see in the next chapter that given a b-divisor $(\phi_{\Pi'})_{\Pi' \in R(\Pi)}$ on Π coming from the geometry of a smooth toroidal compactification, then if for a cofinal subset $S \subseteq R(\Pi)$ we have that the divisor $\phi_{\Pi'}$ belongs to a tropically nef collection $\mathcal{C}$ on Π for all $\Pi' \in S$, then $(\phi_{\Pi'})_{\Pi' \in R(\Pi)}$ automatically defines a continuous tropically nef b-divisor on Π.

We have the following lemma which we will use later on.

Lemma 3.3.6. *Let ϕ_1, ϕ_2 be Cartier divisors on Π such that $\phi_1(v) \geq \phi_2(v)$ for all $v \in |\Pi|$. Then for all positive Minkowski weights $c \in M_1^+(\Pi)$ the inequality*

$$(\phi_1 \cdot c)(0_\Pi) \leq (\phi_2 \cdot c)(0_\Pi)$$

is satisfied.

Proof. We have

$$(\phi_1 \cdot c)(0_\Pi) = \sum_{\sigma \in \Pi(1)} -\phi_1(v_\sigma) c(\sigma) \leq \sum_{\sigma \in \Pi(1)} -\phi_2(v_\sigma) c(\sigma) = (\phi_2 \cdot c)(0_\Pi),$$

as we wanted to show. ■

Definition 3.3.7. Let $c \in M_k(\Pi)$ be a k-dimensional Minkowski weight. The Minkowski weight c is said to be *normalized* if for every $\sigma \in \Pi(k)$, we have that $c(\sigma)$ does not depend on the integral structure on σ.

We now define a normalized version of a Minkowski weight and of the tropical intersection product which is expressed in terms of the metric structure given by the norm $|\cdot|$ on $N_{\mathbb{R}}^{\Pi}$ and does not depend on the integral structure of the conical complex.

Definition 3.3.8. Let $c \in M_k(\Pi)$ be a k-dimensional Minkowski weight. We define the *normalized* Minkowski weight $\hat{c}$ by letting

$$\hat{c}(\sigma) := c(\sigma) \cdot \frac{\sqrt{\det\left(\langle v_\lambda, v_\mu \rangle_{N^\Pi}\right)_{\lambda,\mu \in \sigma(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda, \hat{v}_\mu \rangle_{N^\Pi}\right)_{\lambda,\mu \in \sigma(1)}}},$$

for every $\sigma \in \Pi(k)$, where $\sigma(1)$ denotes the set of rays of σ. Note that $\hat{c}$ is a normalized Minkowski weight in the sense of Definition 3.3.7. Moreover, given a $\mathbb{Q}$-Cartier divisor ϕ on Π we define the *normalized tropical intersection product* $\phi \hat{\odot} c$ by

$$(\phi \hat{\odot} c)(\tau) := \sum_{\substack{\sigma \in \Pi(k) \\ \sigma \geq \tau}} -\phi_\sigma\left(\hat{v}_{\sigma/\tau}\right) c(\sigma) + \phi_\tau \left(\sum_{\substack{\sigma \in \Pi(k) \\ \sigma \geq \tau}} c(\sigma) \hat{v}_{\sigma/\tau} \right),$$

for every $\tau \in \Pi(k-1)$.

We make the following remarks.

Remark 3.3.9. (1) Note that if $c \in M_0(\Pi)$ is 0-dimensional, then $c = \hat{c}$.
(2) Using the normalized tropical intersection product we may compute tropical intersection numbers on a (not necessarily rational) conical complex.

The following proposition allows us to replace the integral structure by the metric structure induced by the norm $|\cdot|$ on $N^{\Pi}_{\mathbb{R}}$.

Proposition 3.3.10. *Let notations be as above. We have the following equality of $\mathbb{R}$-valued Minkowski weights.*

$$\widehat{\phi\cdot c}=\phi\hat{\odot}\hat{c}.$$

Proof. Fix a $\tau\in\Pi(k-1)$. We denote by π_τ the orthogonal projection from $N^{\Pi}_{\mathbb{R}}$ to $N_{\tau\mathbb{R}}^{\Pi}$. Then, since c is a Minkowski weight, recall that

$$\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau}\in N^{\Pi}_{\tau}.$$

Hence, we have that

$$\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau}=\pi_\tau\left(\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau}\right)=\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} c(\sigma)\pi_\tau\left(v_{\sigma/\tau}\right). \tag{3.2}$$

Also, note that the norm of the difference $v_{\sigma/\tau}-\pi_\tau\left(v_{\sigma/\tau}\right)$ is given by

$$\left|v_{\sigma/\tau}-\pi_\tau\left(v_{\sigma/\tau}\right)\right|=\frac{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\sigma(1)}}}{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}. \tag{3.3}$$

Therefore, using Equations (3.2) and (3.3), we get

$$\begin{aligned}
(\widehat{\phi\cdot c})(\tau)&=\left(\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right)c(\sigma)+\phi_\tau\left(\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau}\right)\right)\cdot\frac{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}\\
&=\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} -c(\sigma)\phi_\sigma\left(v_{\sigma/\tau}-\pi_\tau\left(v_{\sigma/\tau}\right)\right)\cdot\frac{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}\\
&=\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} -c(\sigma)\phi_\sigma\left(\left(v_{\sigma/\tau}-\pi_\tau\left(v_{\sigma/\tau}\right)\right)^{\frown}\right)\cdot\frac{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\sigma(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}.
\end{aligned}$$

Now, note that the equality $\left(v_{\sigma/\tau}-\pi_\tau\left(v_{\sigma/\tau}\right)\right)^{\frown}=\left(\hat{v}_{\sigma/\tau}-\pi_\tau\left(\hat{v}_{\sigma/\tau}\right)\right)^{\frown}$ is satisfied. Thus, using Equation (3.3) for $\left|\hat{v}_{\sigma/\tau}-\pi_\tau\left(\hat{v}_{\sigma/\tau}\right)\right|$, we get

$$\begin{aligned}
(\widehat{\phi\cdot c})(\tau)&=\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} -c(\sigma)\phi_\sigma\left(\hat{v}_{\sigma/\tau}-\pi_\tau\left(\hat{v}_{\sigma/\tau}\right)\right)\cdot\frac{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\sigma(1)}}}\cdot\frac{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\sigma(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\tau(1)}}}\\
&=\sum_{\substack{\sigma\in\Pi(k)\\ \sigma\geq\tau}} -c(\sigma)\phi_\sigma\left(\hat{v}_{\sigma/\tau}-\pi_\tau\left(\hat{v}_{\sigma/\tau}\right)\right)\cdot\frac{\sqrt{\det\left(\langle v_\lambda,v_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\sigma(1)}}}{\sqrt{\det\left(\langle \hat{v}_\lambda,\hat{v}_\mu\rangle_{N^\Pi}\right)_{\lambda,\mu\in\sigma(1)}}}\\
&=\left(\phi\hat{\odot}\hat{c}\right)(\tau),
\end{aligned}$$

concluding the proof of the proposition. ∎

Before introducing the notion of the size of a Minkowski weight, we define an auxiliary function $\mathbb{1}_{|\cdot|,\Pi}\colon |\Pi| \to \mathbb{R}$.

Definition 3.3.11. Recall that $\iota_\Pi\colon |\Pi| \to N_{\mathbb{R}}^{\Pi}$ denotes the weak embedding of the conical complex Π. We define the function

$$\mathbb{1}^0_{|\cdot|,\Pi}\colon N_{\mathbb{R}}^{\Pi} \longrightarrow \mathbb{R}$$

by setting

$$\mathbb{1}^0_{|\cdot|,\Pi}(v) = \inf \left\{ f(v) \;\middle|\; \begin{array}{l} \bullet f\colon N_{\mathbb{R}}^{\Pi} \to \mathbb{R} \text{ conical, concave} \\ \bullet f(\iota_\Pi(v_\tau)) = -|v_\tau|,\ \forall \tau \in \Pi(1) \\ \bullet f(v) = 0,\ \forall v \in \iota_\Pi(|\Pi|)^\perp \end{array} \right\}$$

and we let $\mathbb{1}_{|\cdot|,\Pi}\colon |\Pi| \to \mathbb{R}$ be the function defined by

$$\mathbb{1}_{|\cdot|,\Pi} \coloneqq \mathbb{1}^0_{|\cdot|,\Pi} \circ \iota_\Pi.$$

We make some remarks.

Remark 3.3.12. (1) The function $\mathbb{1}^0_{|\cdot|,\Pi}$, being defined as an infimum is concave. Hence, since the composition of a linear function and a concave function is concave, we conclude that the restriction $\mathbb{1}_{|\cdot|,\Pi}|_\sigma$ to every cone $\sigma \in \Pi$ is concave.
(2) Suppose that the function $\tilde{\mathbb{1}}_{|\cdot|,\Pi}\colon |\Pi| \to \mathbb{R}$ given by

$$\tilde{\mathbb{1}}_{|\cdot|,\Pi}(v_\tau) \coloneqq -\left|\iota_\Pi(v_\tau)\right|$$

for $\tau \in \Pi(1)$ and extended by linearity is the restriction of a concave function on $N_{\mathbb{R}}^{\Pi}$. Then

$$\tilde{\mathbb{1}}_{|\cdot|,\Pi} = \mathbb{1}_{|\cdot|,\Pi}.$$

The function $\tilde{\mathbb{1}}_{|\cdot|,\Pi}$ is always the restriction of a concave function on $N_{\mathbb{R}}^{\Pi}$ in dimensions $0,1,2$. For dimensions greater than or equal to 3 this is no longer true because of the existence of flips and flops. Indeed, note that in Figure 3.2, the function $\tilde{\mathbb{1}}_{|\cdot|,\Pi}$ cannot be the restriction of a concave function on $N_{\mathbb{R}}^{\Pi}$.

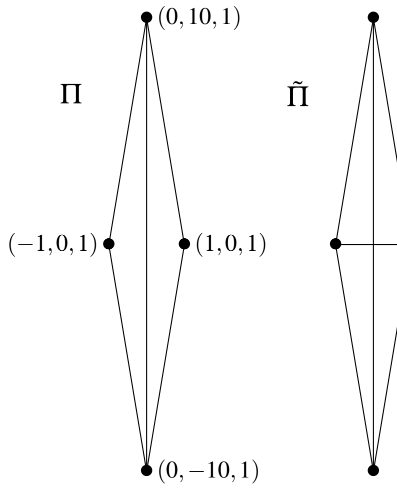


Figure 3.2: 3-dimensional fan cut by the plane $(z = 1)$

(3) The function $\mathbb{1}_{|\cdot|,\Pi}$ is not necessarily piecewise linear with respect to Π, but it is piecewise linear on a (not necessarily rational) refinement $\tilde{\Pi}$ of Π. In Figure 3.2, note that $\mathbb{1}_{|\cdot|,\Pi}$ is piecewise linear on the subdivision $\tilde{\Pi}$. Hence, we may view $\mathbb{1}_{|\cdot|,\Pi}$ as an $\mathbb{R}$-Cartier divisor on $\tilde{\Pi}$ and for every Minkowski weight $c \in M_k(\tilde{\Pi})$ we may consider the normalized tropical product $\mathbb{1}_{|\cdot|,\Pi}\hat{\odot}[\hat{c}] \in Z_{k-1}(\tilde{\Pi})_{\mathbb{R}}$ (see Remark 3.2.18 and part (2) of Remark 3.3.9).

We have the following two lemmas.

Lemma 3.3.13. *Let $\Pi' \geq \Pi$ be a subdivision in $R(\Pi)$. Then the inequality*

$$\mathbb{1}_{|\cdot|,\Pi'} \geq \mathbb{1}_{|\cdot|,\Pi}$$

is satisfied.

Proof. This follows directly from the concavity property of the function $\mathbb{1}_{|\cdot|,\Pi}$. ∎

Lemma 3.3.14. *Let $c \in M_k^+(\Pi)$ be a positive Minkowski weight, then the tropical cycle $\mathbb{1}_{|\cdot|,\Pi}\hat{\odot}[\hat{c}]$ is positive.*

Proof. The tropical cycle $\mathbb{1}_{|\cdot|,\Pi}\hat{\odot}[\hat{c}]$ is well defined by part (2) of Remark 3.3.12. Positivity follows form the concavity of $\mathbb{1}_{|\cdot|,\Pi}$. ∎

We are now ready to define the size of a Minkowski weight.

Definition 3.3.15. Let $\tilde{\Pi}$ be the (not necessarily rational) refinement of Π such that the function $\mathbb{1}_{|\cdot|,\Pi}$ is piecewise linear with respect to $\tilde{\Pi}$. The *size of a k-dimensional, positive Minkowski weight* c in $M_k^+(\Pi)$, denoted by $|c|$, is defined by

$$|c| := \left(\mathbb{1}_{|\cdot|,\Pi}\hat{\odot}\right)^k \left[\widehat{g(c)}\right] \in Z_0(\tilde{\Pi})_{\mathbb{R}} \simeq \mathbb{R},$$

where $g\colon M_k(\Pi) \to M_k(\tilde{\Pi})$ is the morphism of k-dimensional Minkwoski weights given as in 3.1. The *size of a k-dimensional, positive tropical cycle $A \in Z_k^+(\Pi)$*, denoted by $|A|$, is the size of any positive Minkowski weight representing A.

Remark 3.3.16. Note that even though the function $\mathbb{1}_{|\cdot|,\Pi}$ depends on the integral structure of Π, by part (1) of Remark 3.3.9 and Proposition 3.3.10, the size $|\cdot|$ only depends on the metric structure given on $N_{\mathbb{R}}^{\Pi}$.

Example 3.3.17. Assume that $\dim(\Pi) = n \leq 2$ so that $\tilde{\mathbb{1}}_{|\cdot|,\Pi} = \mathbb{1}_{|\cdot|,\Pi'}$ (see part (1) of Remark 3.3.12).

(1) If $c \in M_0^+(\Pi)$ is 0-dimensional, then we have that

$$|c| = \hat{c}\,(0_\Pi) = c\,(0_\Pi) \in \mathbb{Q}_{\geq 0}. \tag{3.4}$$

(2) If $c \in M_1^+(\Pi)$ is 1-dimensional, then we have that

$$\begin{aligned} |c| &= \mathbb{1}_{|\cdot|,\Pi}\hat{\odot}[\hat{c}] = \left[\widehat{\mathbb{1}_{|\cdot|,\Pi}\cdot c}\right] = \mathbb{1}_{|\cdot|,\Pi}\cdot c \\ &= \sum_{\tau\in\Pi(1)} -\mathbb{1}_{|\cdot|,\Pi}(v_\tau)c(\tau) = \sum_{\tau\in\Pi(1)} c(\tau)\,|v_\tau|\,. \end{aligned} \tag{3.5}$$

(3) If $c \in M_2^+(\Pi)$ is 2-dimensional, then we have that

$$|c| = \mathbb{1}_{|\cdot|,\Pi} \hat{\odot} \mathbb{1}_{|\cdot|,\Pi} \hat{\odot} \hat{c} = \left[\mathbb{1}_{|\cdot|,\widehat{\Pi \cdot \mathbb{1}_{|\cdot|,}}\Pi \cdot c\right] = \mathbb{1}_{|\cdot|,\Pi} \cdot \mathbb{1}_{|\cdot|,\Pi} \cdot c = \sum_{\tau \in \Pi(1)} \left(\mathbb{1}_{|\cdot|,\Pi} \cdot c\right)(\tau) \cdot |v_\tau|$$

$$= \sum_{\tau \in \Pi(1)} |v_\tau| \left(\sum_{\substack{\sigma \in \Pi(2) \\ \sigma \geq \tau}} -\mathbb{1}_{|\cdot|,\Pi}\left(v_{\sigma/\tau}\right) c(\sigma) + \mathbb{1}_{|\cdot|,\Pi}\left(\sum_{\substack{\sigma \in \Pi(2) \\ \sigma \geq \tau}} c(\sigma) v_{\sigma/\tau} \right) \right)$$

$$= \sum_{\tau \in \Pi(1)} |v_\tau| \sum_{\substack{\sigma \in \Pi(2) \\ \sigma \geq \tau}} c(\sigma) \left|v_{\sigma/\tau}\right| - \sum_{\substack{\sigma \in \Pi(2) \\ \sigma \geq \tau}} c(\sigma) \left\langle v_{\sigma/\tau}, v_\tau \right\rangle_{N^\Pi}.$$

Now, for each 2-dimensional cone σ, we denote by σ_1 and σ_2 its primitive generators. Since each cone σ has two primitive generators, we can rewrite the above sum as a sum over the 2-dimensional cones in the following way:

$$|c| = \sum_{\tau \in \Pi(1)} |v_\tau| \sum_{\substack{\sigma \in \Pi(2) \\ \sigma \geq \tau}} c(\sigma) \left|v_{\sigma/\tau}\right| - \sum_{\substack{\sigma \in \Pi(2) \\ \sigma \geq \tau}} c(\sigma) \left\langle v_{\sigma/\tau}, v_\tau \right\rangle_{N^\Pi}$$

$$= \sum_{\sigma \in \Pi(2)} 2c(\sigma) \left(|v_{\sigma_1}| \cdot |v_{\sigma_2}| - \langle v_{\sigma_1}, v_{\sigma_2} \rangle_{N^\Pi}\right).$$

In dimensions 3 and higher we cannot give a formula for the size because here we do not necessarily have $\tilde{\mathbb{1}}_{|\cdot|,\Pi} = \mathbb{1}_{|\cdot|,\Pi'}$ (see part (1) of Remark 3.3.12).

In the 0- and 1-dimensional case, we use Equations (3.4) and (3.5) to extend the definition of the size to a (not necessarily positive) 0- and 1-dimensional Minkowski weight, respectively.

Definition 3.3.18. Let c be a (not necessarily positive) Minkowski weight in $M_0(\Pi)$. We define the *size of* c, denoted by $|c|$, to be the non-negative rational number given by

$$|c| := |c(0_\Pi)| \in \mathbb{Q}_{\geq 0}.$$

If c is a (not necessarily positive) Minkowski weight in $M_1(\Pi)$, then we define the *size of* c, denoted by $|c|$, to be the non-negative real number given by

$$|c| := \sum_{\tau \in \Pi(1)} |c(\tau)| \, |v_\tau|.$$

The following is a key lemma.

Lemma 3.3.19. *Let $c \in Z_*^+(\Pi)$ be a k-dimensional positive tropical cycle. Then, if $\phi \in \mathrm{Div}(\Pi)_{\mathbb{Q}}$ is such that either $\phi \cdot c$ is a positive tropical cycle, or such that $\phi \cdot c$ is a (not necessarily positive)* 0- *or* 1-*dimensional tropical cycle, then the inequality*

$$|\phi \cdot c| \leq \left(\sup_{\hat{v} \in \mathbb{S}^\Pi} |\phi(\hat{v})| \right) \cdot |c|$$

is satisfied.

Proof. First, suppose that $\phi \cdot c$ is a positive tropical cycle. Let $\Pi' \geq \Pi$ be a (not necessarily rational) refinement such that $\mathbb{1}_{|\cdot|,\Pi}$ is piecewise linear in Π'. We define the positive real constant B by

$$B := \sup_{\tau \in \Pi'(1)} |\phi\left(\hat{v}_\tau\right)|.$$

Then, for every $\tau \in \Pi'(1)$ we have that

$$\phi(v_\tau) = |v_\tau|\,\phi(\hat{v}_\tau) = -|v_\tau|\,(-\phi(\hat{v}_\tau)) \geq -|v_\tau|B \geq \mathbb{1}_{|\cdot|,\Pi}(v_\tau)B.$$

Hence, since both ϕ and $\mathbb{1}_{|\cdot|,\Pi}\cdot B$ are piecewise linear in Π', we conclude that

$$\phi \geq \mathbb{1}_{|\cdot|,\Pi}B.$$

Therefore, using Lemma 3.3.6 and the positivity of $\left(\mathbb{1}_{|\cdot|,\Pi}\,\hat{\odot}\right)^{k-1}[\hat{c}]$, we get

$$|\phi\cdot c| = \left(\mathbb{1}_{|\cdot|,\Pi}\,\hat{\odot}\right)^{k-1}\phi\cdot[\hat{c}] = \phi\cdot\left(\mathbb{1}_{|\cdot|,\Pi}\,\hat{\odot}\right)^{k-1}[\hat{c}] \leq B\left(\mathbb{1}_{|\cdot|,\Pi}\,\hat{\odot}\right)^{k}[\hat{c}] = B|c|,$$

as we wanted to show.

Now, suppose that $\phi\cdot c$ is a (not necessarily positive) 0-dimensional tropical cycle. We will use the explicit formula for the size of 1-dimensional Minkowski weights given in Example 3.3.17. We get

$$\begin{aligned}
|\phi\cdot c| &= |\phi\cdot c\,(0_\Pi)| \\
&= \left|\sum_{\tau\in\Pi(1)} -\phi_\tau(v_\tau)c(\tau) + \phi_\tau\left(\sum_{\tau\in\Pi(1)} c(\tau)v_\tau\right)\right| \\
&= \left|\sum_{\tau\in\Pi(1)} -\phi_\tau(v_\tau)c(\tau)\right| \\
&= \left|\sum_{\tau\in\Pi(1)} -\phi_\tau(\hat{v}_\tau)c(\tau)|v_\tau|\right| \\
&\leq \left(\sup_{\hat{v}\in\mathbb{S}^\Pi}|\phi(\hat{v})|\right)\cdot|c|,
\end{aligned}$$

where the third equality follows since c is a Minkowski weight.

Finally, we consider the case that $\phi\cdot c$ is a (not necessarily positive) 1-dimensional tropical cycle. We will use the explicit formulas for the size of 1- and 2-dimensional Minkowski weights given in Example 3.3.17. Assume that $c\in M_2(\Pi)$ and that $\phi\cdot c\in M_1(\Pi)$. We have that

$$\begin{aligned}
|\phi\cdot c| &= \sum_{\tau\in\Pi(1)} |\phi\cdot c(\tau)|\cdot|v_\tau| \\
&= \sum_{\tau\in\Pi(1)}\left|\sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right)c(\sigma) + \phi_\tau\left(\sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau}\right)\right|\cdot|v_\tau|.
\end{aligned}$$

Since c is a Minkowski weight, we have that

$$\sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau} = a_\tau v_\tau$$

for some $a_\tau\in\mathbb{Q}$. Taking the dot product with v_τ on both sides, we get the expression

$$a_\tau = \sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} \frac{c(\sigma)\langle v_{\sigma/\tau}, v_\tau\rangle_{N^\Pi}}{\langle v_\tau, v_\tau\rangle_{N^\Pi}}.$$

Hence, we have the sequence of inequalities

$$\begin{aligned}|\phi \cdot c| &= \sum_{\tau\in\Pi(1)} \left| \sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right)c(\sigma) + \phi_\tau\left(\sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} c(\sigma)v_{\sigma/\tau}\right)\right| \cdot |v_\tau| \\ &= \sum_{\tau\in\Pi(1)} \left| \sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right)c(\sigma) + a_\tau\phi_\tau\left(v_\tau\right)\right| \cdot |v_\tau| \\ &= \sum_{\tau\in\Pi(1)} \left| \sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right)c(\sigma) + \frac{c(\sigma)\langle v_{\sigma/\tau}, v_\tau\rangle_{N^\Pi}}{\langle v_\tau, v_\tau\rangle_{N^\Pi}}\phi_\tau\left(v_\tau\right)\right| \cdot |v_\tau|.\end{aligned}$$

Now, as before, for each 2-dimensional cone σ, we denote by σ_1 and σ_2 its primitive generators. Since each cone σ has two primitive generators, we can rewrite the above sum as a sum over the 2-dimensional cones in the following way:

$$\begin{aligned}|\phi \cdot c| &= \sum_{\tau\in\Pi(1)} \left| \sum_{\substack{\sigma\in\Pi(2)\\ \sigma\geq\tau}} -\phi_\sigma\left(v_{\sigma/\tau}\right)c(\sigma) + \frac{c(\sigma)\langle v_{\sigma/\tau}, v_\tau\rangle_{N^\Pi}}{\langle v_\tau, v_\tau\rangle_{N^\Pi}}\phi_\tau\left(v_\tau\right)\right| \cdot |v_\tau| \\ &= \sum_{\sigma\in\Pi(2)} \Bigg| -\phi_\sigma\left(v_{\sigma_1}\right)c(\sigma)\left|v_{\sigma_2}\right| + \frac{c(\sigma)\langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}}{\langle v_{\sigma_2}, v_{\sigma_2}\rangle_{N^\Pi}}\left|v_{\sigma_2}\right|\phi_{\sigma_2}\left(v_{\sigma_2}\right) - \phi_\sigma\left(v_{\sigma_2}\right)c(\sigma)\left|v_{\sigma_1}\right| \\ &\qquad + \frac{c(\sigma)\langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}}{\langle v_{\sigma_1}, v_{\sigma_1}\rangle_{N^\Pi}}\left|v_{\sigma_1}\right|\phi_{\sigma_1}\left(v_{\sigma_1}\right)\Bigg| \\ &= \sum_{\sigma\in\Pi(2)} \Big| -\phi_\sigma\left(\hat{v}_{\sigma_1}\right)c(\sigma)\left|v_{\sigma_1}\right|\left|v_{\sigma_2}\right| + c(\sigma)\langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}\phi_{\sigma_2}\left(\hat{v}_{\sigma_2}\right) - \phi_\sigma\left(\hat{v}_{\sigma_2}\right)c(\sigma)\left|v_{\sigma_1}\right|\left|v_{\sigma_2}\right| \\ &\qquad + c(\sigma)\langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}\phi_{\sigma_1}\left(\hat{v}_{\sigma_1}\right)\Big| \\ &= \sum_{\sigma\in\Pi(2)} \Big| -\phi_\sigma\left(\hat{v}_{\sigma_1}\right)c(\sigma)\left(\left|v_{\sigma_1}\right|\left|v_{\sigma_2}\right| - \langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}\right) - \phi_\sigma\left(\hat{v}_{\sigma_2}\right)c(\sigma)\left(\left|v_{\sigma_1}\right|\left|v_{\sigma_2}\right| - \langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}\right)\Big| \\ &= \sum_{\sigma\in\Pi(2)} c(\sigma)\left(\left|v_{\sigma_1}\right|\left|v_{\sigma_2}\right| - \langle v_{\sigma_1}, v_{\sigma_2}\rangle_{N^\Pi}\right)\left|-\phi_\sigma\left(\hat{v}_{\sigma_1}\right) - \phi_\sigma\left(\hat{v}_{\sigma_2}\right)\right| \\ &\leq \left(\sup_{\hat{v}\in\mathbb{S}^\Pi}|\phi(\hat{v})|\right)\cdot|c|,\end{aligned}$$

thus concluding the proof of the lemma. ■

The following two important propositions are consequences of Lemma 3.3.19.

Proposition 3.3.20. *Let $\mathcal{C}$ be a tropically nef collection on Π. Then for every $\mathbb{Q}$-Cartier divisor ϕ in $\mathcal{C}$, the measure μ_ϕ from Definition 3.2.23 is bounded, i.e. the quantity*

$$\left|\int_{\mathbb{S}^\Pi} g(u)\mu_\phi\right|$$

is bounded for every continuous, bounded function g on $\mathbb{S}^\Pi$.

Proof. Let ϕ be a $\mathbb{Q}$-Cartier divisor in $\mathcal{C}$ defined on $\Pi' \geq \Pi \in R(\Pi)$ and let g be a continuous, bounded function on $\mathbb{S}^{\Pi}$. Let $B := \sup_{v\in\mathbb{S}^{\Pi}} |g(v)|$. Then, using Lemma 3.3.19 and noting that $\left(\phi^{n-1}\cdot[\Pi']\right)(\tau) \geq 0$ for $\tau \in \Pi'(1)$, we get the sequence of inequalities

$$\begin{aligned}
\left|\int_{\mathbb{S}^{\Pi}} g(u)\,\mu_{\phi}\right| &= \left|\sum_{\tau\in\Pi'(1)} g\left(\hat{v}_{\tau}\right)\left(\phi^{n-1}\cdot[\Pi']\right)(\tau)\,|v_{\tau}|\right| \\
&\leq \sum_{\tau\in\Pi'(1)} \left|g\left(\hat{v}_{\tau}\right)\right|\left(\phi^{n-1}\cdot[\Pi']\right)(\tau)\,|v_{\tau}| \\
&\leq B\left|\phi^{n-1}\cdot[\Pi']\right| \\
&\leq B\left(\sup_{v\in\mathbb{S}^{\Pi}}|\phi(v)|\right)^{n-1}\cdot\left|[\Pi']\right|,
\end{aligned}$$

concluding the proof of the proposition. ■

The following definition is a particular case of [AL06, Definition 4.2.5 and Proposition 4.2.5].

Definition 3.3.21. Let $\Xi := \left\{\mu \,|\, \mu \text{ is a finite measure on } \mathbb{S}^{\Pi}\right\}$. There exists a norm $||\cdot||$ on Ξ, called the *total variation norm*, defined for every $\mu \in \Xi$ by

$$||\mu|| := \sup\left\{\sum_{i=1}^{\infty}|\mu(A_i)| \,\middle|\, \{A_i\}_{i\geq 1}\subseteq\mathbb{S}^{\Pi} \text{ measurable}, A_i\cap A_j=\emptyset \text{ for } i\neq j, \bigcup_{i\geq 1}A_i=\mathbb{S}^{\Pi}\right\}.$$

Proposition 3.3.22. *Consider a continuous tropically nef b-divisor $(\phi_{\Pi'})_{\Pi'\in R(\Pi)}$ on Π. In particular we have that its restriction to $\mathbb{S}^{\Pi}$ is uniformly Cauchy. Then the net of discrete measures $\left\{\mu_{\phi_{\Pi'}}\right\}_{\Pi'\in R(\Pi)}$ is Cauchy as well (with respect to the total variaton norm).*

Proof. Let $\Pi'' \geq \Pi'$ in $R'(\Pi)$. First, note that we may view $\phi_{\Pi'}^{n-1}\cdot[\Pi']$ as a Minkowski weight in $M_1(\Pi'')$ simply by putting 0 on the rays in $\Pi''(1)\setminus\Pi'(1)$. Moreover, the piecewise linearity of $\phi_{\Pi'}$ implies that for such a ray τ, we have

$$\phi_{\Pi'}^{n-1}\cdot[\Pi'']\,(v_{\tau}) = 0.$$

This implies that we can write

$$\phi_{\Pi'}^{n-1}\cdot[\Pi'] = \phi_{\Pi'}^{n-1}\cdot[\Pi'']$$

as an equality of Minkowski weights on Π''. Moreover, combining Lemma 3.3.13 and Lemma 3.3.6 we have that

$$\left|[\Pi']\right| \geq \left|[\Pi'']\right|.$$

We define the real positive constants C, D and E by

$$C := \sup_{v\in\mathbb{S}^{\Pi}}|\phi_{\Pi'}(v)|, \quad D := \sup_{v\in\mathbb{S}^{\Pi}}|\phi_{\Pi''}(v)|, \quad E := |[\Pi]|.$$

We can write

$$\begin{aligned}
||\mu_{\phi_{\Pi'}} - \mu_{\phi_{\Pi''}}|| &= \left|\left| \sum_{\tau\in\Pi'(1)} \left(\phi_{\Pi'}^{n-1}\cdot[\Pi']\right)(\tau)|v_\tau|\,\delta_{\hat{v}_\tau} - \sum_{\tau\in\Pi''(1)} \left(\phi_{\Pi''}^{n-1}\cdot[\Pi'']\right)(\tau)|v_\tau|\,\delta_{\hat{v}_\tau}\right|\right| \\
&= \left|\left| \sum_{\tau\in\Pi''(1)} \left(\phi_{\Pi'}^{n-1}\cdot[\Pi'']\right)(\tau)|v_\tau|\,\delta_{\hat{v}_\tau} - \sum_{\tau\in\Pi''(1)} \left(\phi_{\Pi''}^{n-1}\cdot[\Pi'']\right)(\tau)|v_\tau|\,\delta_{\hat{v}_\tau}\right|\right| \\
&= \left|\left| \sum_{\tau\in\Pi''(1)} \left(\phi_{\Pi'}^{n-1}-\phi_{\Pi''}^{n-1}\right)[\Pi''](\tau)\,|v_\tau|\,\delta_{\hat{v}_\tau}\right|\right| \\
&= \sum_{\tau\in\Pi''(1)} \left|\left(\phi_{\Pi'}^{n-1}-\phi_{\Pi''}^{n-1}\right)[\Pi''](\tau)\right||v_\tau|.
\end{aligned}$$

Note that

$$\sum_{\tau\in\Pi''(1)} \left|\left(\phi_{\Pi'}^{n-1}-\phi_{\Pi''}^{n-1}\right)[\Pi''](\tau)\right||v_\tau| = \left|\left(\phi_{\Pi'}^{n-1}-\phi_{\Pi''}^{n-1}\right)[\Pi'']\right|,$$

where the $|\cdot|$ on the right hand side denotes the size of the Minkowski weight. Hence, using Lemma 3.3.19, we can bound the total variation norm of the difference $\mu_{\Pi'} - \mu_{\Pi''}$ by

$$\begin{aligned}
||\mu_{\phi_{\Pi'}} - \mu_{\phi_{\Pi''}}|| &\le \left|\left(\phi_{\Pi'}^{n-1}-\phi_{\Pi''}^{n-1}\right)[\Pi'']\right| \\
&= \left|(\phi_{\Pi'}-\phi_{\Pi''})\sum_{\ell=0}^{n-2}\left(\phi_{\Pi'}^{\ell}\phi_{\Pi''}^{n-2-\ell}\cdot[\Pi'']\right)\right| \\
&\le \sup_{v\in\mathbb{S}^\Pi}|\phi_{\Pi'}(v)-\phi_{\Pi''}(v)|\left|\sum_{\ell=0}^{n-2}\left(\phi_{\Pi'}^{\ell}\phi_{\Pi''}^{n-2-\ell}\cdot[\Pi'']\right)\right| \\
&\le \sup_{v\in\mathbb{S}^\Pi}|\phi_{\Pi'}(v)-\phi_{\Pi''}(v)|\sum_{\ell=0}^{n-2}C^{\ell}D^{n-2-\ell}E,
\end{aligned}$$

as we wanted to show. ■

The following definition is adapted from [AL06, Definition 9.3.2].

Definition 3.3.23. Let $\{\mu_{\Pi'}\}_{\Pi'\in R(\Pi)}$ be a net of bounded, discrete measures on $\mathbb{S}^\Pi$. The net converges weakly to a limit measure μ on $\mathbb{S}^\Pi$ if for every bounded, continuous function g on $\mathbb{S}^\Pi$, we have that

$$\lim_{\Pi'\in R(\Pi)}\int_{\mathbb{S}^\Pi} g(u)\mu_{\Pi'} = \int_{\mathbb{S}^\Pi} g(u)\mu.$$

We now state one of the main results of this chapter.

Theorem 3.3.24. *Let $\phi = (\phi_{\Pi'})_{\Pi'\in R(\Pi)}$ be a continuous tropically nef b-divisor on Π. Then the net of discrete measures $\mu_{\phi_{\Pi'}}$ weakly converges to a finite measure μ_ϕ on $\mathbb{S}^\Pi$, called the* limit measure.

Proof. For every $\Pi'\in R(\Pi)$, the discrete measure $\mu_{\phi_{\Pi'}}$ is bounded by Proposition 3.3.20. The existence of the limit measure μ_ϕ follows from Proposition 3.3.22, since the space $(\Xi, ||\cdot||)$ is complete ([AL06, Proposition 4.2.6]). Finally the fact that

$$\lim_{\Pi'\in R(\Pi)}\int_{\mathbb{S}^\Pi} g(u)\mu_{\phi_{\Pi'}} = \int_{\mathbb{S}^\Pi} g(u)\mu_\phi$$

is clear, since g can be uniformly approximated by simple functions, i.e. finite linear combinations of indicator functions of measurable sets, and μ_ϕ arises as a limit in total variation of the discrete measures $\mu_{\phi_{\Pi'}}$ for $\Pi'\in R(\Pi)$. ■

Corollary 3.3.25. *Let ϕ be a continuous tropically nef b-divisor on Π as above. Then the limit (in the sense of nets) defined by*

$$\phi^n := \lim_{\Pi' \in R(\Pi)} \phi^n_{\Pi'} \cdot [\Pi']$$

exists, is finite and is given by

$$\phi^n = \int_{\mathbb{S}^\Pi} \phi(u) \mu_\phi.$$

The quantity ϕ^n is called the degree *of the continuous tropically nef b-divisor ϕ on Π.*

Remark 3.3.26. Note that by definition of a tropically nef collection $\mathcal{C}$ on Π, we have that for every natural number ℓ and for every collection of continuous tropically nef b-divisors $\phi_1, \dots, \phi_\ell$, the linear combination $\lambda_1 \phi_1 + \cdots + \lambda_\ell \phi_\ell$ is a continuous tropically nef b-divisor on Π for every choice of non-negative real numbers $\lambda_1, \dots, \lambda_\ell$.

Proposition/Definition 3.3.27. *There is a symmetric map from the space of $(n-1)$-tuples of continuous tropically nef b-divisors on Π to the space of finite measures on $\mathbb{S}^\Pi$, called the* mixed limit measure*, such that for every natural number ℓ and for every choice of non-negative real numbers $\lambda_1, \dots, \lambda_\ell$, the equality*

$$\mu_{\lambda_1\phi_1+\ldots+\lambda_\ell\phi_\ell} = \sum_{i_1,\dots,i_{n-1}=1}^{\ell} \lambda_{i_1} \dots \lambda_{i_\ell} \mu_{\phi_{i_1},\dots,\phi_{i_{n-1}}}$$

is satisfied for every collection $\phi_1, \dots, \phi_\ell$ of continuous tropically nef b-divisors on Π.

The following corollary follows from the definition of the mixed measure and Corollary 3.3.25.

Corollary 3.3.28. *Let $\phi_1, \dots, \phi_n$ and $\mu_{\phi_1}, \dots, \mu_{\phi_n}$ as above. Let $\left\{\phi_{i,\Pi'}\right\}_{\Pi' \in R(\Pi)}$ be the family of $\mathbb{Q}$-Cartier divisors converging uniformly to ϕ_i on $\mathbb{S}^\Pi$ for $i = 1, \dots, n$. Then the limit (in terms of nets) defined by*

$$\phi_1 \cdots \phi_n := \lim_{\Pi' \in R(\Pi)} \phi_{1,\Pi'} \cdots \phi_{n,\Pi'} \cdot [\Pi']$$

exists, is finite and is given by

$$\phi_1 \cdots \phi_n = \int_{\mathbb{S}^\Pi} \phi_1(u) \mu_{\phi_2,\dots,\phi_n}.$$

Moreover, for any $1 \leq i \leq n$, we have integral formulae

$$\int_{\mathbb{S}^\Pi} \phi_1(u) \mu_{\phi_2,\dots,\phi_n} = \int_{\mathbb{S}^\Pi} \phi_i(u) \mu_{\phi_1,\dots,\hat{\phi}_i,\dots,\phi_n}.$$

The quantity $\phi_1 \cdots \phi_n$ is called the mixed degree *of the continuous tropically nef b-divisors $\phi_1, \dots, \phi_n$.*

We end this section by giving a formula for computing the difference $\phi_1^n - \phi_2^n$, where ϕ_1, ϕ_2 are two continuous tropically nef b-divisors on Π.

Corollary 3.3.29. *Let ϕ_1 and ϕ_2 be two continuous tropically nef b-divisors on Π. Then we can compute the difference $\phi_1^n - \phi_2^n$ in the following way:*

$$\phi_1^n - \phi_2^n = \sum_{i=0}^{n-1} \int_{\mathbb{S}^\Pi} \left(\phi_1(u) - \phi_2(u)\right) \mu_{\underbrace{\phi_1, \dots, \phi_1}_{i\text{-times}}, \underbrace{\phi_2, \dots, \phi_2}_{(n-1-i)\text{-times}}}.$$

Proof. This is a straightforward computation using Corollary 3.3.28. ∎

3.4 Surface area measures

Throughout this section Π will denote a balanced, cp, n-dimensional weakly embedded conical complex with a fixed metric structure $|\cdot|$ on $N_{\mathbb{R}}^{\Pi}$. The goal of this section is to identify the limit measure μ_ϕ associated to a continuous tropically nef b-divisor ϕ on Π, defined in the previous section, as a generalization of the so called surface area measure associated to a bounded, compact, convex body (which we will see can be seen as induced from tropical intersection theory on a fan) to arbitrary weakly embedded conical complexes. The original works of Aleksandrov ([Ale37a, Ale37b, Ale38a, Ale38b, Ale39]) and of Fenchel and Jessen ([FJ38]) are recommended references for surface area measures and related concepts. We however follow the more recent and thorough survey by Schneider ([Sch93]).

Suppose that $\Pi = \Sigma \subseteq N_{\mathbb{R}}$ is a smooth, complete fan. Then any concave function ϕ on $N_{\mathbb{R}}$ can be seen as a continuous tropically nef b-divisor on Π. The results about the relationship of volumes of bounded, compact, convex sets and degrees of nef toric b-divisors in Chapter 1 can be derived directly from the results of this section. Moreover, if we assume that ϕ is of class C^2, then we can give explicit formulas for computing integrals with respect to these limit measures (and the mixed version thereof) in terms of Lebesgue integrals of determinants of Hessians (Proposition 3.4.16). More generally, if we assume smoothness and concavity in the interior of every cone of the fan and existence of directional derivatives along its boundary, then, by decomposing the conical complex Π into the disjoint union of the interiors of its cones, we can express the limit measure μ_ϕ as a sum of Lebesgue measures (Corollary 3.4.18). This result leads to formulating Conjecture 3.4.20 in the general case of a conical complex which is not necessarly a fan. In the 2-dimensional case, however, we can give a formula for the difference of degrees of two particular continuous tropically nef b-divisors (Proposition 3.4.22).

We start with some definitions.

Definition 3.4.1. Consider the vector space $\mathbb{R}^n$ equipped with standard euclidean metric. For $0 \leq k \leq n$, we let $\mathcal{H}^k$ be the k-dimensional *Hausdorff measure* on $\mathbb{R}^n$. In particular, if ω is a Borel subset of a k-dimensional euclidean space E^k or a k-dimensional sphere $\mathbb{S}^k$ in $\mathbb{R}^n$, then $\mathcal{H}^k(\omega)$ coincides with the k-dimensional *Lebesgue* measure of ω computed in E^k or with the k-dimensional *spherical Lebesgue* measure of ω computed in $\mathbb{S}^k$, respectively.

Let $K \subseteq \mathbb{R}^n$ be an n-dimensional bounded, compact, convex set and let $h_K\colon \mathbb{R}^n \to \mathbb{R}$ be its corresponding *support function*, defined by the assignment

$$v \longmapsto \inf_{m\in K} \langle m, v\rangle.$$

It is a concave, conical function. Moreover, let

$$g_K\colon \mathbb{S}^{n-1} \longrightarrow \mathcal{P}(\partial K), \tag{3.6}$$

where $\mathcal{P}(\partial K)$ denotes the power set of the boundary ∂K of K, be the map given in the following way. In the case that h_K is of class C^2, g_K sends $u \in \mathbb{S}^{n-1}$ to the gradient $\nabla h_K(u)$. In general, the inverse g^{-1} is what in the literature is called the *Gauss* map, which assigns the outer unit normal vector $v_K(x)$ to an $x \in \partial_* K$, where $\partial_* K$ consists of all points in the boundary ∂K of K having a unique outer normal vector. In other words, we have that

$$g_K(u) = \left\{ m \in \mathbb{R}^n \,\middle|\, \langle m,u\rangle = h_K(u) \text{ and } \langle m, v\rangle \geq h_K(v), \forall v \in \mathbb{R}^n \right\},$$

for every $u \in \mathbb{S}^{n-1}$. This is also the map given by the Legendre–Fenchel duality (see Section 3.5).

Definition 3.4.2. The *surface area measure* $S_{n-1}(K,\cdot)$ of K is the finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$ defined by

$$S_{n-1}(K,\omega) = \mathcal{H}^{n-1}\left(g_K(\omega)\right)$$

for every Borel subset ω of $\mathbb{S}^{n-1}$.
In particular, for a polytope P with unitary normal vectors $u_1,\dots,u_r$ at its facets $F_1,\dots,F_r$, respectively, the surface area measure of a Borel subset $\omega \subseteq \mathbb{S}^{n-1}$ is given by

$$S_{n-1}(P,\omega) = \sum_{u_i\in\omega} \mathrm{vol}_{n-1}(F_i),$$

where vol_k denotes the k-dimensional volume operator. In other words, we have that

$$S_{n-1}(P,\cdot) = \sum_{i=1}^{r} \mathrm{vol}_{n-1}(F_i)\,\delta_{u_i},$$

where δ_{u_i} denotes the Dirac delta measure supported on $u_i \subset \mathbb{S}^{n-1}$.

Example 3.4.3. Let P be a polytope and h_P its corresponding piecewise linear, concave support function. We get the formula

$$n\,\mathrm{vol}_n(P) = \sum_{i=1}^{r} h_P(u_i)\mathrm{vol}_{n-1}(F_i) = \int_{\mathbb{S}^{n-1}} h_P(u)S_{n-1}(P,u)$$

for the volume of P. This formula can be generalized to any full dimensional bounded, compact, convex set in the following way: for every n-dimensional bounded, compact, convex set K, the volume of K is given by

$$\mathrm{vol}_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u)S_{n-1}(K,u). \tag{3.7}$$

Remark 3.4.4. Using the same notation as above, we have by [Sch93, Lemma 5.1.1] that in the case of a polytope P, the relation

$$\sum_{i=1}^{r} h_P(u_i)\mathrm{vol}_{n-1}(F_i)\,u_i = 0$$

is satisfied. This is not a coincidence as the values $\mathrm{vol}_{n-1}(F_i)$ correspond to the weights of the Minkowski weight $h_P^{n-1}\cdot[\Sigma_P] \in M_1(\Sigma_P)$, where $\Sigma_P \subseteq N_{\mathbb{R}}$ is the normal fan of P and we have identified $N_{\mathbb{R}}$ and $\mathbb{R}^n$ (see the proof of Proposition 3.4.5).

The case of a fan

We fix a lattice N of rank n and assume that $\Pi = \Sigma \subseteq N_{\mathbb{R}}$ is a smooth, complete fan so that $|\cdot|$ is an euclidean metric on $N_{\mathbb{R}}^{\Sigma} = N_{\mathbb{R}}$ coming from an inner product which we denote by $\langle,\rangle_N$. Moreover, we fix an identification $N_{\mathbb{R}} \simeq \mathbb{R}^n$ and we view $|\cdot|$ as the standard euclidean metric on $\mathbb{R}^n$. Consider the support function h_K of an n-dimensional bounded, compact, convex set $K \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$. Its restriction to the sphere can be obtained as a uniform limit of piecewise linear, concave functions on $N_{\mathbb{R}}$. This can be done since the space of piecewise linear functions on a compact set is dense with respect to uniform convergence, and by concavity of the limit function, the piecewise linear functions can be chosen to be concave as well. Hence, h_K defines a continuous tropically nef b-divisor on Σ. Using Remark 3.2.24, we may consider the measure μ_{h_K} as a Borel measure on the sphere $\mathbb{S}^{n-1}$. The measures μ_{h_K} and $S_{n-1}(K,\cdot)$ are related by the following proposition.

Proposition 3.4.5. *With notations as above, the measures $S_{n-1}(K,\cdot)$ and μ_{h_K} are related by the formula*

$$(n-1)!\,S_{n-1}(K,\omega)=\mu_{h_K}(\omega)$$

for every Borel subset $\omega\subseteq\mathbb{S}^{n-1}$.

Proof. Since the measures $S_{n-1}(K,\cdot)$ and μ_{h_K} are defined as a weak limit of discrete measures corresponding to polytopes and piecewise linear functions, respectively, it suffices to consider a rational polytope $K=P$ with piecewise linear, rational, concave support function h_P. As before, we denote by $u_1,\dots,u_r$ the unitary normal vectors at the facets $F_1,\dots,F_r$ of P, respectively. Furthermore, consider the toric divisor D_P associated to P defined in Section 1.1 and let τ_i be the ray spanned by u_i. Let $h_P(\tau_i)$ be the support function on the star $\Pi(\tau_i)$ as defined in Lemma 3.2.17. Then we have that

$$\left(h_P^{n-1}\cdot[\Sigma]\right)(\tau_i)=(h_P(\tau_i))^{n-1}\cdot[\Sigma(\tau_i)]=\left(D_P|_{V(\tau_i)}\right)^{n-1}=(n-1)!\,\mathrm{vol}(F_i),$$

where the first equality follows from Lemma 3.2.17 and the others are standard toric geometrical facts, in particular, $D_P|_{V(\tau_i)}$ denotes the restriction of the toric divisor D_P to the toric subvariety $V(\tau_i)$. The statement of the proposition now easily follows. ■

Example 3.4.3 and the above proposition imply the following expression for the volume of a bounded, compact, convex set $K\subseteq M_{\mathbb{R}}$ in terms of the limit measure μ_{h_K}.

Lemma 3.4.6. *Let $K\subseteq M_{\mathbb{R}}$ be a bounded, compact, convex set with support function h_K. Then the volume of K is given by*

$$\mathrm{vol}_n(K)=\frac{1}{n!}\int_{\mathbb{S}^{n-1}}h_K(u)\mu_{h_K}. \tag{3.8}$$

Remark 3.4.7. The above formulas (3.7) and (3.8) for the volume of a bounded, compact, convex set are also related to what in the literature is called the *cone-volume measure* of a convex body.

Remark 3.4.8. In the 2-dimensional case we can be more explicit. Indeed, let θ be the coordinate on the 1-dimensional sphere $\mathbb{S}^1$. Then the differential $S_1(K,\theta)$ is the same as $\partial_x^2\tilde{h}_K(\theta)\mathrm{d}\theta$, where ∂_x^2 is the second derivative in the tangential direction x and

$$\tilde{h}_K\colon[0,2\pi]\longrightarrow\mathbb{R}$$

is the function defined by $\tilde{h}_K(\theta)=h_K(u_\theta)$ for $u_\theta=e_1\cos(\theta)+e_2\sin(\theta)$, with (e_1,e_2) the standard orthonormal basis of $\mathbb{R}^2$. Note that we are not assuming smoothness, so we view these derivatives in the sense of distributions.

Example 3.4.9. Consider the concave support function $h_K\colon\mathbb{R}^2\to\mathbb{R}$ given by

$$h_K(a,b)=\begin{cases}\frac{ab}{a+b}\,, & a,b\in\mathbb{R}_{\geq 0},\\ \min\{0,a,b\}\,, & \text{otherwise.}\end{cases}$$

The corresponding bounded, compact, convex set $K\subseteq\mathbb{R}^2$ is

$$K=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,x,y\geq 0,\quad x+y\leq 1,\quad \sqrt{x}+\sqrt{y}\geq 1\right\},$$

(see Example 1.4.7). The measure $\partial_x^2\tilde{h}_K(\theta)\mathrm{d}\theta$ is concentrated at the point $p=\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and along the interval $0<\theta<\pi/2$ (see Figure 3.3). Note that in principle, it could be supported at the points $(1,0)$ and $(0,1)$ as well. However, the derivative of h_K restricted to a tangent line at these points is continuous, hence the image of g_K at these points is 0-dimensional and thus the measure μ_{h_K} is not supported here.

For $0<\theta<\pi/2$, the second partial derivative in the tangential direction $x=(-\sin(\theta),\cos(\theta))$ is given by

$$\begin{aligned}\partial_x^2\tilde{h}_K(\theta)\mathrm{d}\theta &= (-\sin(\theta),\cos(\theta))\,\mathrm{Hess}\left(\tilde{h}_K\left(\cos(\theta),\sin(\theta)\right)\right)\begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix}\mathrm{d}\theta\\ &= \frac{-2}{\left(\cos(\theta)+\sin(\theta)\right)^3}\mathrm{d}\theta.\end{aligned}$$

Here, $\mathrm{Hess}\left(\tilde{h}_K\left(\cos(\theta),\sin(\theta)\right)\right)$ denotes the Hessian matrix of $\tilde{h}_K$.

For the point p, a tangent vector at p is given by $x=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and a tangent line at p is given by $\ell_p\colon\left(-\frac{1}{\sqrt{2}}+t\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}-t\frac{1}{\sqrt{2}}\right)$. Hence, we have

$$\begin{aligned}\tilde{h}_K\left(p+tx\right) &= \tilde{h}_K|_{\ell_p}(t)\\ &= \min\left(-\frac{1}{\sqrt{2}}+t\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}-t\frac{1}{\sqrt{2}}\right)\\ &= -\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}|t|.\end{aligned}$$

Taking the second derivative (in terms of distributions), we get

$$\partial_x^2\tilde{h}_K(\theta)\mathrm{d}\theta|_p = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{h}_K\left(p+tx\right)\bigg|_{t=0} = -\frac{1}{\sqrt{2}}\cdot 2\delta_0 = -\sqrt{2}\delta_0,$$

where δ_0 refers to the Dirac delta distribution.

Hence, the volume of K is given by

$$\begin{aligned}2\,\mathrm{vol}_2(K) &= \int_{\mathbb{S}^1}\tilde{h}_K(\theta)S_{n-1}(K,\mathrm{d}\theta)\\ &= \int_{\mathbb{S}^1}\tilde{h}_K(\theta)\partial_x^2\tilde{h}_K(\theta)\mathrm{d}\theta\\ &= -\sqrt{2}h_K\left(p\right)+\int_0^{\pi/2}h_K\left(\cos(\theta),\sin(\theta)\right)\frac{-2}{\left(\cos(\theta)+\sin(\theta)\right)^3}\mathrm{d}\theta\\ &= 1-2\int_0^{\pi/2}\frac{\cos(\theta)\sin(\theta)}{\left(\cos(\theta)+\sin(\theta)\right)^4}\mathrm{d}\theta\\ &= 1-\frac{1}{3}=\frac{2}{3},\end{aligned}$$

and thus $\mathrm{vol}_2(K)=1/3$.

Remark 3.4.10. As can be seen in the example above, the support of the measure μ_{h_K} can be decomposed into a smooth part (the interval $0<\theta<\pi/2$), where it is given as a Lebesgue measure using the Hessian of the class C^2 function $h_K|_{(0,\pi/2)}$, and a discrete part (the point p). We will see that this is usually the case.

Definition 3.4.11. We define the set $\mathcal{K}_n$ to be the set of all n-dimensional bounded, compact, convex bodies in $M_{\mathbb{R}}\simeq\mathbb{R}^n$.

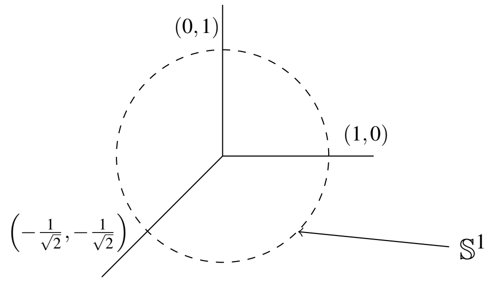


Figure 3.3: Calculating volumes with the surface area measure

As is described in [Sch93, Section 5], one can generalize the *surface area measure* associated to a bounded, compact convex body to a collection of n (not necessarily distinct) bounded, compact convex bodies. This is the so called *mixed surface area measure*. The following is [Sch93, Theorem 5.1.7].

Theorem/Definition 3.4.12. *There is a nonnegative symmetric function* $\mathrm{V}\colon\ (\mathcal{K}_n)^n \to \mathbb{R}$*, called the* mixed volume*, such that for every natural number* ℓ *and for every non-negative real numbers* $\lambda_1,\dots,\lambda_\ell$*, the equation*

$$\mathrm{vol}_n(\lambda_1 K_1+\cdots+\lambda_\ell K_\ell)=\sum_{i_1,\dots,i_n=1}^{\ell}\lambda_{i_1}\cdots\lambda_{i_n}\mathrm{V}\left(K_{i_1},\dots,K_{i_n}\right),$$

where the sum on the left hand side is the Minkowski sum of convex sets, is satisfied for any collection of convex bodies $K_1,\dots,K_\ell\in\mathcal{K}_n$*.*

Furthermore, there is a symmetric map S *from* $(\mathcal{K}_n)^{n-1}$ *into the space of finite Borel measures on* $\mathbb{S}^{n-1}$*, called the* mixed surface area measure*, such that for every natural number* ℓ *and for every non-negative real numbers* $\lambda_1,\dots,\lambda_\ell$*, the equation*

$$S_{n-1}\left(\lambda_1K_1+\cdots+\lambda_\ell K_\ell,\omega\right)=\sum_{i_1,\dots,i_{n-1}=1}^{\ell}\lambda_{i_1}\cdots\lambda_{i_{n-1}}S\left(K_{i_1},\dots,K_{i_{n-1}},\omega\right)$$

is satisfied for $K_1,\dots,K_\ell\in\mathcal{K}_n$ *and for every Borel subset* $\omega\subseteq\mathbb{S}^{n-1}$*. Moreover, for* $K_1,\dots,K_n\in\mathcal{K}_n$*, the mixed volume* $\mathrm{V}\left(K_1,\dots,K_n\right)$ *can be expressed in terms of the mixed surface area measure in the following way*

$$\mathrm{V}\left(K_1,\dots,K_n\right)=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h_{K_1}(u)S\left(K_2,\dots,K_n,u\right).$$

We make the following remarks.

Remark 3.4.13. (1) Setting $K=K_1=\cdots=K_n$ we get

$$\begin{aligned}\mathrm{vol}_n(K)&=\mathrm{V}(K,\dots,K)\\&=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h_K(u)S\left(K,\dots,K,u\right)\\&=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h_K(u)S_{n-1}\left(K,u\right)\\&=\frac{(n-1)!}{n!}\int_{\mathbb{S}^{n-1}}h_K(u)S_{n-1}\left(K,u\right)\\&=\frac{1}{n!}\int_{\mathbb{S}^{n-1}}h_K(u)\mu_{h_k},\end{aligned}$$

as in Equation (3.8).
(2) The mixed volume "V$(\cdot)$" defined above is related to the Mixed Volume "MV$(\cdot)$" defined in Section 1.1 by the formula

$$\mathrm{V}(K_1,\ldots,K_n) = \frac{1}{n!}\mathrm{MV}(K_1,\ldots,K_n)$$

for $K_1,\ldots,K_n \in \mathcal{K}_n$.

The following proposition relating the mixed surface area measure and the mixed limit measure from Definition 3.3.27 follows directly from the definitions.

Proposition 3.4.14. *Let $K_1,\ldots,K_{n-1}$ be convex bodies in $\mathcal{K}_n$ with support functions $h_1,\ldots,h_{n-1}$, respectively. Then the mixed surface area measure $S(K_1,\ldots,K_{n-1},\cdot)$ and the mixed limit measure $\mu_{h_1,\ldots,h_{n-1}}$ from Definition 3.3.27 (as a measure on $\mathbb{S}^{n-1}$) are related by*

$$(n-1)!\,S(K_1,\ldots,K_{n-1},\omega) = \mu_{h_1,\ldots,h_{n-1}}(\omega)$$

for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$.

Let $K_1,\ldots,K_n$ be a collection of convex bodies in $\mathcal{K}_n$. In the case that the support functions $h_{K_1},\ldots,h_{K_n}$ are of class C^2, Proposition 3.4.16 below gives us an explicit way of computing integrals with respect to the mixed surface area measure in terms of Lebesgue integrals of determinants of Hessians. Before stating the proposition, we make a definition. This is taken from [Sch93, pg. 119].

Definition 3.4.15. Let K be a convex body in $\mathcal{K}_n$ and assume that its support function $h = h_K$ is of class C^2. For $j \in \{1,\ldots,n-1\}$ and for $u \in \mathbb{R}^n$ we define the numbers $s_j(K,u)$ in such a way that $\binom{n-1}{j}s_j(K,u)$ is equal to the sum of the principal minors of order j of the Hessian matrix $(h_{ij})_{i,j=1}^n$ of h at u with respect to any orthonormal basis of $\mathbb{R}^n$. In particular, for $j=1$, we have

$$s_1(K,u) = \frac{1}{n-1}\Delta h(u),$$

where Δ denotes the Laplace operator on $\mathbb{R}^n$.

If the orthonormal basis $(e_1,\ldots,e_n)$ of $\mathbb{R}^n$ is chosen such that $e_n = u$, then

$$s_{n-1}(K,u) = \det\left(h_{ij}(u)\right)_{i,j=1}^{n-1}.$$

Moreover, for $K_1,\ldots,K_{n-1} \in \mathcal{K}_n$, the function

$$s(K_1,\ldots,K_{n-1},\cdot)\colon \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$

is defined by the fact that for every natural number ℓ and for every $K_1,\ldots,K_\ell \in \mathcal{K}_n$, the equation

$$s_{n-1}(\lambda_1 K_1 + \cdots + \lambda_\ell K_\ell, u) = \sum_{i_1,\ldots,i_{n-1}=1}^{\ell} \lambda_{i_1}\cdots\lambda_{i_{n-1}} s\left(K_{i_1},\ldots,K_{i_{n-1}},u\right),$$

is satisfied for every positive real numbers $\lambda_1,\ldots,\lambda_\ell$, and the fact that s is symmetric in its $n-1$ arguments.

Proposition 3.4.16. *Let $K_1, \dots, K_n$ be a collection of convex bodies in $\mathcal{K}_n$ with support functions $h_1, \dots, h_n$ of class C^2. Then the mixed limit measure $\mu_{h_1,\dots,h_{n-1}}$ is given by*

$$\frac{1}{(n-1)!}\mu_{h_1,\dots,h_{n-1}}(\omega) = S(K_1, \dots, K_{n-1}, \omega) = \int_\omega s(K_1, \dots, K_{n-1}, u)\,\mathrm{d}u$$

for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$.
The mixed volume $V(K_1, \dots, K_n)$ of $K_1, \dots, K_n$ now reads

$$\mathrm{V}(K_1, \dots, K_n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_1(u) s(K_2, \dots, K_n, u)\,\mathrm{d}u.$$

Moreover, since the mixed volume is symmetric in its arguments, we have integral formulae such as

$$\int_{\mathbb{S}^{n-1}} h_1(u) s(K_2, K_3, \dots, K_n, u)\,\mathrm{d}u = \int_{\mathbb{S}^{n-1}} h_2(u) s(K_1, K_3, \dots, K_n, u)\,\mathrm{d}u.$$

In particular, if $K = K_1 = \dots = K_n$, then the volume of K can be expressed in terms of a Lebesgue integral by

$$\mathrm{vol}_n(K) = V(K, \dots, K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u) s_{n-1}(K, \dots, K, u)\mathrm{d}u.$$

Proof. This is a combination of [Sch93, Corollary 2.5.3], the results in [Sch93, Section 5.3] and Proposition 3.4.14. ∎

We now look at the following more general case. Let $K \in \mathcal{K}_n$ and suppose that its support function h_K satisfies the following properties:

(1) The restriction $h_K|_{\mathrm{int}(\sigma)}$ to the interior of each cone $\sigma \in \Pi$ is of class C^2.

(2) All directional derivatives of h_K exist in the boundary $\partial\sigma$ for all cones $\sigma \in \Pi$.

Let $\sigma \in \Pi(d)$ be a d-dimensional cone. Our goal is to identify the restricted measure

$$\mu_{h_K}|_{\mathrm{int}(\sigma)}.$$

For this, let K_σ be any bounded, compact, convex body in $\mathbb{R}^d$ such that the restriction of its support function to σ is h_{K_σ}, i.e. such that $h_{K_\sigma}|_\sigma = h_K|_\sigma$. By abuse of notation we denote by $\mu_{h_K|_\sigma}$ the measure on $\mathbb{S}^{n-1} \cap \mathrm{int}(\sigma)$ associated to the restricted function $h_K|_{\mathrm{int}(\sigma)}$. We decided not to denote it by $\mu_{h_K|_{\mathrm{int}(\sigma)}}$ so that it doesn't get mixed up with the restricted measure $\mu_{h_K}|_{\mathrm{int}(\sigma)}$. By Propositions 3.4.5 and 3.4.16 we have that

$$\mu_{h_K|_\sigma}(\omega \cap \mathrm{int}(\sigma)) = (d-1)!\int_{\omega\cap\mathrm{int}(\sigma)} s_{d-1}(K_\sigma, u)$$

for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$. The following theorem identifies the restricted measure $\mu_{h_K}|_{\mathrm{int}(\sigma)}$. Before stating it, let us fix some notation. We will denote by $N_{\sigma\mathbb{R}} \subseteq N_\mathbb{R}$ the linear subspace generated by σ with the induced metric and we let $N(\sigma)_\mathbb{R} = N_\mathbb{R}/N_{\sigma\mathbb{R}}$ be the quotient space. Recall that the star $\Pi(\sigma)$ of Π at σ is a complete fan in $N(\sigma)_\mathbb{R}$. Moreover, we identify $N(\sigma)_\mathbb{R}$ with the orthogonal complement $N_{\sigma\mathbb{R}}^{\perp}$ via the map induced by $\langle,\rangle_N$, i.e. via $v + \sigma \mapsto (n \mapsto \langle v, n\rangle_N)$.

Theorem 3.4.17. *Let notations and assumptions be as above. We write Δ_f for the stability set of a concave, conical function f (see Section 1.1). Then for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$, we have that*

$$\mu_{h_K}(\omega \cap \operatorname{int}(\sigma)) = \frac{(n-1)!}{(d-1)!} \int_{\omega \cap \operatorname{int}(\sigma)} \operatorname{vol}_{n-d}\left(\Delta_{\tilde{h}_{K,p}}\right) \mu_{h_K|_\sigma},$$

where for $p \in \mathbb{S}^{n-1} \cap \operatorname{int}(\sigma)$, the function

$$\tilde{h}_{K,p} \colon N(\sigma)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is a concave, conical function given by the assignment

$$w \longmapsto \sup_{v \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{v+w} h_K(p) - \mathrm{D}_v h_K(p)\right),$$

where D_x denotes the directional derivative of h_K at p in the direction of x, i.e.

$$\mathrm{D}_x h_K(p) = \frac{\mathrm{d}}{\mathrm{d}t} h_K(p+tx)|_{t=0}.$$

Proof. First we show that $\tilde{h}_{K,p}$ is conical and concave. Let λ be a positive real number. We have that

$$\begin{aligned}
\tilde{h}_{K,p}(\lambda w) &= \sup_{v \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{v+\lambda w} h_K(p) - \mathrm{D}_v h_K(p)\right) \\
&= \sup_{v \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{\lambda v+\lambda w} h_K(p) - \mathrm{D}_{\lambda v} h_K(p)\right) \\
&= \sup_{v \in N_{\sigma\mathbb{R}}} \lambda \left(\mathrm{D}_{v+w} h_K(p) - \mathrm{D}_v h_K(p)\right) \\
&= \lambda \tilde{h}_{K,p}(w),
\end{aligned}$$

and $\tilde{h}_K$ is thus conical. To prove concavity, it suffices to prove that

$$\tilde{h}_{K,p}(w_1+w_2) \geq \tilde{h}_{K,p}(w_1) + \tilde{h}_{K,p}(w_2),$$

for any $w_1, w_2 \in N(\sigma)_{\mathbb{R}}$. Now, for any $v_1, v_2 \in N_{\sigma\mathbb{R}}$, since $p \in \operatorname{int}(\sigma)$ and $h_K|_{\operatorname{int}(\sigma)}$ is of class C^2, we have that

$$\mathrm{D}_{v_1} h_K(p) + \mathrm{D}_{v_2} h_K(p) = \mathrm{D}_{v_1+v_2} h_K(p).$$

Also, by concavity of h_K we have that

$$\mathrm{D}_{v_1+w_1} h_K(p) + \mathrm{D}_{v_2+w_2} h_K(p) \leq \mathrm{D}_{v_1+v_2+w_1+w_2} h_K(p)$$

for any $w_1, w_2 \in N(\sigma)_{\mathbb{R}}$ and $v_1, v_2 \in N_{\sigma\mathbb{R}}$. Hence, we get that

$$\begin{aligned}
\tilde{h}_{K,p}(w_1) + \tilde{h}_{K,p}(w_2) &= \sup_{v_1, v_2 \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{v_1+w_1} h_K(p) - \mathrm{D}_{v_1} h_K(p) + \mathrm{D}_{v_2+w_2} h_K(p) - \mathrm{D}_{v_2} h_K(p)\right) \\
&\leq \sup_{v_1, v_2 \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{v_1+v_2+w_1+w_2} h_K(p) - \mathrm{D}_{v_1+v_2} h_K(p)\right) \\
&= \sup_{v \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{v+w_1+w_2} h_K(p) - \mathrm{D}_v h_K(p)\right) \\
&= \tilde{h}_{K,p}(w_1+w_2),
\end{aligned}$$

and thus $\tilde{h}_{K,p}$ is concave.

Now, note that the inclusion $N_{\sigma\mathbb{R}} \hookrightarrow N_{\mathbb{R}}$ induces a surjection of the dual spaces $M_{\mathbb{R}} \twoheadrightarrow N_{\sigma\mathbb{R}}^{\vee}$. Since $h_K|_{\mathrm{int}(\sigma)}$ is of class C^2, we have an induced application

$$g_K|_{N_{\sigma\mathbb{R}}} : \mathbb{S}^{n-1} \cap \mathrm{int}(\sigma) \longrightarrow N_{\sigma\mathbb{R}}^{\vee},$$

given by the assignment

$$v \longmapsto \nabla h_K|_{N_{\sigma\mathbb{R}}}(v).$$

We have the following commutative diagram.

$$\begin{tikzcd}
 & M_{\mathbb{R}} \arrow[d, "\pi"] \\
\mathbb{S}^{n-1} \cap \mathrm{int}(\sigma) \arrow[ur, "g_K"] \arrow[r, "g_K|_{N_{\sigma\mathbb{R}}}"'] & N_{\sigma\mathbb{R}}^{\vee}
\end{tikzcd}$$

Figure 3.4: Calculating the restricted measure

We have to understand the fibre of the application π. For $p \in \mathbb{S}^{n-1} \cap \mathrm{int}(\sigma)$ we have

$$\begin{aligned}
\pi^{-1}\left(\nabla h_K|_{N_{\sigma\mathbb{R}}}(p)\right) &= g_K(p) \\
&= \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle m, p\rangle = h_K(p) \text{ and } \langle m, v\rangle \geq h_K(v), \forall v \in N_{\mathbb{R}} \right\} \\
&= \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle \nabla h_K|_{N_{\sigma\mathbb{R}}}(p), v\rangle = \langle m, v\rangle, \forall v \in N_{\sigma\mathbb{R}}, \right. \qquad (3.9) \\
&\qquad \left. \text{and } \langle m, p+w\rangle \geq \tilde{h}_{K,p}(w) + h_K(p), \forall w \in N(\sigma)_{\mathbb{R}} \right\} \\
&= \Delta_{\tilde{h}_{K_p}}.
\end{aligned}$$

To see the equality (3.9), let $m_0 \in \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle m, p\rangle = h_K(p) \text{ and } \langle m, v\rangle \geq h_K(v), \forall v \in N_{\mathbb{R}} \right\}$. In particular, we have that $m_0|_{N_{\sigma\mathbb{R}}} \geq h_K|_{N_{\sigma\mathbb{R}}}$ and that $\langle m_0, p\rangle = h_K(p)$. Note that by the Euler formula for homogeneous functions of weight 1 we have that $\langle \nabla h_K(p), p\rangle = h_K(p)$. Hence, since $h_K|_{\mathrm{int}(\sigma)}$ is of class C^2, this implies that

$$m_0|_{N_{\sigma\mathbb{R}}} = \nabla h_K(p)|_{N_{\sigma\mathbb{R}}},$$

and thus the first condition is satisfied. Now, by hypothesis we have that $\langle m_0, v\rangle \geq h_K(v)$ for all $v \in N_{\mathbb{R}}$. In particular, we have that $\langle m_0, p\rangle \geq h_K(p)$. Hence, by linearity, we get

$$\mathrm{D}_v h_K(p) \leq \mathrm{D}_v \langle m_0, p\rangle.$$

Moreover, by what was said above, we have that

$$\mathrm{D}_v \langle m_0, p\rangle = \mathrm{D}_v \langle \nabla h_K(p), p\rangle = \mathrm{D}_v h_K(p) = \langle \nabla h_K(p), v\rangle = \langle m_0, v\rangle$$

for all $v \in N_{\sigma\mathbb{R}}$. Hence, we get

$$\mathrm{D}_{v+w} h_K(p) - \mathrm{D}_v h_K(p) \leq \langle m_0, v+w\rangle - \langle m_0, v\rangle = \langle m_0, w\rangle.$$

Therefore,

$$\tilde{h}_{K,p}(w) = \sup_{v \in N_{\sigma\mathbb{R}}} \left(\mathrm{D}_{v+w} h_K(p) - \mathrm{D}_v h_K(p)\right) \leq \langle m_0, w\rangle,$$

and hence the second condition is satisfied.

Conversely, suppose that

$$m_0 \in \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle \nabla h_K|_{N_{\sigma\mathbb{R}}}(p), v\rangle = \langle m, v\rangle, \forall v \in N_{\sigma\mathbb{R}}, \text{ and } \langle m, p+w\rangle \geq \tilde{h}_{K,p}(w) + h_K(p), \forall w \in N(\sigma)_{\mathbb{R}} \right\}.$$

Again, using the Euler formula for homogeneous functions of weight 1 we have that

$$\langle m_0, p\rangle = \langle \nabla h_K(p), p\rangle = h_K(p).$$

This proves the first condition. To prove the second condition, let $v \in N_{\mathbb{R}}$. If $v \in N_{\sigma\mathbb{R}}$, then we have that

$$\langle m_0, v\rangle = \langle \nabla h_K(p), v\rangle = \mathrm{D}_v h_K(p).$$

Hence, by concavity of h_K, we get

$$h_K(v) \leq h_K(p) + \mathrm{D}_{v-p} h_K(p) = h_K(p) + \langle m_0, v-p\rangle = \langle m_0, v\rangle,$$

as we wanted to show. If $v \notin N_{\sigma\mathbb{R}}$ we write $v = v_1 + w_1$ with $v_1 \in N_{\sigma\mathbb{R}}$ and $w_1 \in N_{\sigma\mathbb{R}}^{\perp} \simeq N(\sigma)_{\mathbb{R}}$. Then we have that

$$\mathrm{D}_v h_K(p) = \mathrm{D}_{v_1+w_1} h_K(p) \leq \tilde{h}_p(w_1) + \mathrm{D}_{v_1} h_K(p) \leq \langle m_0, w_1\rangle + \langle m_0, v_1\rangle = \langle m_0, v\rangle.$$

Using the concavity of h_K we get

$$h_K(v) \leq h_K(p) + \mathrm{D}_{v-p} h_K(p) \leq h_K(p) + \langle m_0, v-p\rangle = \langle m_0, v\rangle,$$

concluding the proof of the equality (3.9).

Now, recall the commutative diagram (3.4). Since the volume of the image of a Borel subset via g_K can be computed as the volume of the image of the restricted map $g_K|_{N_{\sigma\mathbb{R}}}$ times the volume of the fibre π, it follows that for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$ the equality

$$\mathrm{vol}_{n-1}\left(g_K(\omega \cap \mathrm{int}(\sigma))\right) = \int_{\omega\cap\mathrm{int}(\sigma)} \mathrm{vol}_{n-d}\left(\Delta_{\tilde{h}_{K,p}}\right) \mathrm{vol}_{d-1}\left(g_K|_{N_{\sigma\mathbb{R}}}(\omega \cap \mathrm{int}(\sigma))\right) \mathrm{d}p$$

is satisfied. Using Proposition 3.4.5 and the fact that $\mu_{h_K|_\sigma}$ is a Lebesgue measure, we get

$$\begin{aligned}
\mu_{h_K}(\omega \cap \mathrm{int}(\sigma)) &= (n-1)!\mathcal{H}^{n-1}\left(g_K(\omega \cap \mathrm{int}(\sigma))\right) \\
&= (n-1)!\mathrm{vol}_{n-1}\left(g_K(\omega \cap \mathrm{int}(\sigma))\right) \\
&= \frac{(n-1)!}{(d-1)!} \int_{\omega\cap\mathrm{int}(\sigma)} \mathrm{vol}_{n-d}\left(\Delta_{\tilde{h}_{K_p}}\right) \mu_{h_K|_\sigma},
\end{aligned}$$

concluding the proof of the theorem. ■

We get the following corollary.

Corollary 3.4.18. *Let $K \in \mathcal{K}_n$ and assume that its support function h_K satisfies properties (1) and (2) above. Then for every Borel subset $\omega \subseteq \mathbb{S}^{n-1}$ we have that*

$$\begin{aligned}
\mu_{h_K}(\omega) &= \sum_{\sigma\in\Pi} \mu_{h_K}|_{\mathrm{int}(\sigma)}(\omega \cap \mathrm{int}(\omega)) \\
&= (n-1)! \sum_{\sigma\in\Pi} \int_{\omega\cap\mathrm{int}(\sigma)} \frac{1}{(\dim(\sigma)-1)!} \mathrm{vol}_{n-\dim(\sigma)}\left(\Delta_{\tilde{h}_{K,p}}\right) \mu_{h_K|_\sigma},
\end{aligned}$$

where the measure $\mu_{h_K|_\sigma}$ and the function $\tilde{h}_{K_p}$ are the ones given above for each σ.

Example 3.4.19. Let h_K be as in Example 3.4.9. It is easy to check that h_K satisfies properties (1) and (2) above.

The only 2-dimensional cone on which the measure μ_{h_K} is non-zero is $\sigma \simeq \mathbb{R}_{\geq 0}$. We start by describing the restricted measure $\mu_{h_K}|_{\mathrm{int}(\sigma)}$. For this, consider the orthonormal basis (e_1, e_2) with

$$e_1 = (\cos(\theta), \sin(\theta)) \quad \text{and} \quad e_2 = (-\sin(\theta), \cos(\theta)).$$

Let $u = e_2$. Then, for $0 < \theta < \pi/2$, the term $s(K,u) = s_1(K,u) = h_{11}(u)$ is the upper left entry of the matrix

$$J \cdot \mathrm{Hess}\,(h(u)) \cdot J^{\intercal},$$

where $J = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is the Jacobian matrix of the transformation from euclidean into polar coordinates. One checks that

$$h_{11}(u) = \frac{-2}{(\cos(\theta) + \sin(\theta))^3}$$

for $0 < \theta < \pi/2$. Hence, we have that

$$\mu_{h_K}(\omega \cap \mathrm{int}(\sigma)) = \int_{\omega \cap \mathrm{int}(\sigma)} \frac{-2}{(\cos(\theta) + \sin(\theta))^3} \mathrm{d}\theta$$

for every Borel subset $\omega \subseteq \mathbb{S}^1$.

On the other hand, let $\tau = \mathbb{R}_{\geq 0}(-1,-1)$ and consider $p = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. The orthogonal subspace $N_{\tau\mathbb{R}}^{\perp}$ is given by $\left\{(\lambda, -\lambda) \,\middle|\, \lambda \in \mathbb{R}\right\}$. The function $\tilde{h}_{K,p} \colon N(\tau)_{\mathbb{R}} \simeq N_{\tau\mathbb{R}}^{\perp} \to \mathbb{R}$ is given by the assignment

$$\lambda \longmapsto \begin{cases} \lambda & \text{if } \lambda \geq 0, \\ -\lambda & \text{if } \lambda \leq 0. \end{cases}$$

Thus, $\Delta_{\tilde{h}_{K,p}}$ is the set given by

$$\left\{ m \in \left(N_{\sigma\mathbb{R}}^{\perp}\right)^{\vee} \,\middle|\, m \geq -1 \text{ and } -m \geq -1 \right\} \simeq \left\{(0,1) + t(1,0) \,\middle|\, 0 \leq t \leq 1\right\} \subset \mathbb{R}^2$$

and

$$\mathrm{vol}_1\left(\Delta_{\tilde{h}_{K,p}}\right) = \sqrt{2}.$$

Combining the above, we get

$$\begin{aligned} 2\mathrm{vol}_2(K) &= \int_{\mathbb{S}^1} h_K(u) \mu_{h_K} = \sum_{\sigma \in \Pi} \int_{\mathbb{S}^1 \cap \mathrm{int}(\sigma)} h_K(u) \mu_{h_K}|_{\mathrm{int}(\sigma)} \\ &= -\sqrt{2} h_K(p) + \int_0^{\pi/2} h_K(\cos(\theta), \sin(\theta)) \frac{-2}{(\cos(\theta) + \sin(\theta))^3} \mathrm{d}\theta \\ &= 1 - 2 \int_0^{\pi/2} \frac{\cos(\theta)\sin(\theta)}{(\cos(\theta) + \sin(\theta))^4} \mathrm{d}\theta \\ &= 1 - \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

This coincides with the computations done in Example 3.4.9.

The general case

We would like to generalize Theorem 3.4.17 (and hence Corollary 3.4.18) to the case of a smooth, balanced, cp, weakly embedded conical complex Π with a fixed metric structure on $N_{\mathbb{R}}^{\Pi}$. Note that the problem here is that Proposition 3.4.5 is no longer true in this situation. We however formulate the following conjecture.

Conjecture 3.4.20. *Let $\phi\colon |\Pi| \to \mathbb{R}$ be a continuous tropically nef b-divisor on Π satisfying the following properties:*

(1) The restriction $\phi_K|_\sigma$ to each cone $\sigma \in \Pi$ is of class C^2.

(2) All directional derivatives of ϕ exist in the boundary $\partial\sigma$ for all cones $\sigma \in \Pi$.

Let $N_{\sigma\,\mathbb{R}}^{\Pi} \subseteq N_{\mathbb{R}}^{\Pi}$ be the linear subspace generated by $\iota_\Pi(\sigma)$ with the induced metric. We write $\operatorname{int}(\sigma)$ although we actually mean $\operatorname{int}(\iota_\Pi(\sigma))$. Also, consider the star $N(\sigma)_{\mathbb{R}}^{\Pi(\sigma)} \subseteq N_{\sigma\,\mathbb{R}}^{\Pi\,\perp}$ with the induced metric. (Note that in contrast to the fan situation, we do not have an identification $N(\sigma)_{\mathbb{R}}^{\Pi(\sigma)} \simeq N_{\sigma\,\mathbb{R}}^{\Pi\,\perp}$.) Moreover, let $\mu_{\phi|_\sigma}$ be the measure associated to the restricted function $\phi|_{\operatorname{int}(\sigma)}$. Then, for every Borel subset $\omega \subseteq \mathbb{S}^{\Pi}$, we have that

$$\mu_\phi\left(\omega \cap \operatorname{int}(\sigma)\right) = \frac{(n-1)!}{(\dim(\sigma)-1)!}\int_{\omega\cap\operatorname{int}(\sigma)} \operatorname{vol}_{n-\dim(\sigma)}\left(\Delta_{\tilde{\phi}_p}\right)\mu_{\phi|_\sigma},$$

where for $p \in \mathbb{S}^{\Pi} \cap \operatorname{int}(\sigma)$, the function

$$\tilde{\phi}_p\colon N(\sigma)_{\mathbb{R}}^{\Pi(\sigma)} \longrightarrow \mathbb{R}$$

is a concave, conical function given by the assignment

$$w \longmapsto \sup_{v\in N_{\sigma\,\mathbb{R}}^{\Pi}} \left(\mathrm{D}_{v+w}\,\phi(p) - \mathrm{D}_v\,\phi(p)\right),$$

where D_x denotes the directional derivative of ϕ at p in the direction of x.

In the case of dimension 2 we can say something more. Suppose that we have a continuous tropically nef b-divisor ϕ on a 2-dimensional smooth, cp, balanced conical complex Π. We write ϕ^{Π} for the piecewise linear function induced by ϕ on Π. Moreover, we let σ be a 2-dimensional cone. By smoothness, we may assume that $\iota_\Pi(\sigma) = \mathbb{R}^2_{\geq 0}$. Let b be a sufficiently negative integer number so that the linear extension of both ϕ_σ and ϕ_σ^{Π} to the whole of $\mathbb{R}^2$ taking value b at $e_0 = (-1,-1)$ is concave. We denote these extensions again by ϕ_σ and ϕ_σ^{Π}.

Definition 3.4.21. With notations as above we define the bounded, compact, convex sets

$$K_\Pi^\sigma(\phi) := \Delta_{\phi_\sigma^\Pi} \quad \text{and} \quad K^\sigma(\phi) := \Delta_{\phi_\sigma}.$$

Note that $K_\Pi^\sigma(\phi)$ is a polygon, whereas $K^\sigma(\phi)$ is not necessarily so.

Proposition 3.4.22. *Let notations be as above. The difference of degrees $\left(\phi^{\Pi}\right)^2 - \phi^2$ satisfies the relation*

$$\left(\phi^{\Pi}\right)^2 - \phi^2 = \sum_{\sigma\in\Pi(2)} c_\sigma,$$

where the c_σ's are given by

$$c_\sigma = \int_{\mathrm{relint}(\sigma)\cap\mathbb{S}^1} \left(\phi_\sigma^\Pi(u) - \phi_\sigma(u)\right) S\left(K^\sigma(\phi), u\right) + \int_{\mathrm{relint}(\sigma)\cap\mathbb{S}^1} \left(\phi_\sigma^\Pi(u) - \phi_\sigma(u)\right) S\left(K_\Pi^\sigma(\phi), u\right).$$

In particular, in the polyhedral case, we can compute the c_σ's as follows: let $\tilde{\Pi}$ be a sufficiently large smooth subdivision of Π such that ϕ is piecewise linear with respect to $\tilde{\Pi}$. For every ray $\tau \in \tilde{\Pi}(1)$ choose affine functions ψ^Π and ψ on N_σ such that $\phi^\Pi|_\tau = \psi^\Pi|_\tau$ and $\phi|_\tau = \psi|_\tau$. Then $\phi^\Pi - \psi^\Pi$ and $\phi - \psi$ induce functions on $\tilde{\Pi}(\tau)$ which are integral linear on each face. We denote these by $\phi^\Pi(\tau)$ and by $\phi(\tau)$, respectively. Then we can express c_σ by the following expression

$$c_\sigma = \sum_{\substack{\tau\in\mathrm{relint}(\sigma)\\ \tau\in\tilde{\Pi}(1)}} \left(\phi_\sigma^\Pi(v_\tau) - \phi_\sigma(v_\tau)\right)\left(\phi^\Pi(\tau) + \phi(\tau)\right)\cdot[\tilde{\Pi}(\tau)],$$

where the " $\cdot$" denotes the tropical intersection product.

Proof. For a full-dimensional cone $\sigma \in \Pi(2)$ we have

$$\mu_\phi\left(\omega \cap \mathrm{int}(\sigma)\right) = (n-1)!S_{n-1}(K^\sigma(\phi), \omega).$$

Since ϕ and ϕ^Π only differ along the interiors of the full-dimensional cones, we have that

$$\begin{aligned}
\left(\phi^\Pi\right)^2 - \phi^2 &= \sum_{\sigma\in\Pi(2)} \sum_{i=0}^{2-1} (2-1)! \int_{\mathbb{S}^\Pi\cap\mathrm{int}(\sigma)} \left(\phi_\sigma(u) - \phi_\sigma^\Pi(u)\right) \mu_{\underbrace{\phi,\dots,\phi}_{i\text{-times}},\underbrace{\phi^\Pi,\dots,\phi^\Pi}_{(2-1-i)\text{-times}}} \\
&= \sum_{\sigma\in\Pi(2)} \sum_{i=0}^{1} \int_{\mathbb{S}^\Pi\cap\mathrm{int}(\sigma)} \left(\phi_\sigma(u) - \phi_\sigma^\Pi(u)\right) S(\underbrace{K_\Pi^\sigma(\phi),\dots,K_\Pi^\sigma(\phi)}_{i\text{-times}},\underbrace{K^\sigma(\phi),\dots,K^\sigma(\phi)}_{(1-i)\text{-times}},u).
\end{aligned}$$

To see the polyhedral case, note that we can write

$$\begin{aligned}
(\phi_\Pi)^2 - \phi^2 &= \left(\phi^\Pi - \phi\right)\left(\phi^\Pi + \phi\right) \\
&= \sum_{\tau\in\tilde{\Pi}(1)} \left(\phi^\Pi - \phi\right)\left(\phi^\Pi(\tau) + \phi(\tau)\right)\cdot[\tilde{\Pi}(\tau)] \\
&= \sum_{\sigma\in\Pi(2)} \sum_{\substack{\tau\in\mathrm{relint}(\sigma)\\ \tau\in\tilde{\Pi}(1)}} \left(\phi_\sigma^\Pi(v_\tau) - \phi_\sigma(v_\tau)\right)\left(\phi^\Pi(\tau) + \phi(\tau)\right)\cdot[\tilde{\Pi}(\tau)],
\end{aligned}$$

as we wanted to prove. ■

3.5 Canonical decomposition of a difference of convex sets

Throughout this section, N will denote a lattice of rank n and $M = N^\vee$ will denote its dual lattice. The goal of this section is to show that given two full-dimensional convex sets $K_1 \subseteq K_2 \subseteq M_\mathbb{R}$ there is a canonical decomposition of the difference $K_2 \setminus K_1$ and to interpret the volume of the pieces geometrically in terms of intersection numbers. We will start by recalling the Legendre–Fenchel duality for convex sets. Most of the definitions and statements which we will state regarding this duality can be found in [Roc72]. We also refer to [BPS14, Chapter 2]. Then, we will define what it means for two faces, each one belonging to one of the convex sets, to be related. Using this relationship, we will show that the complement of a full-dimensional convex set contained in another convex set canonically decomposes into a disjoint union of convex sets, indexed over the faces of the larger set. Finally, using the mixed surface area measures defined in the previous section we give a geometric interpretation of this canonical decomposition in terms of tropical intersection numbers, in the case that K_2 is polyhedral (Proposition 3.5.27).

Legendre–Fenchel duality

We start with some definitions.

Definition 3.5.1. Let K be a convex set in $M_{\mathbb{R}}$. A convex subset $F \subseteq K$ is called a *face* of K if, for every closed line segment $[m_1, m_2] \subseteq K$ such that $\operatorname{relint}([m_1, m_2]) \cap F \neq \emptyset$, the inclusion $[m_1, m_2] \subseteq F$ holds. A non-empty subset $F \subseteq K$ is called an *exposed face* of K if there exists an $v \in N_{\mathbb{R}}$ such that

$$F = \left\{ m \in K \,\middle|\, \langle v, m \rangle = \min_{m' \in K} \langle v, m' \rangle \right\}.$$

Remark 3.5.2. Every exposed face is a face. However, not every face is exposed, as can be seen in the figure 3.5. Here, the star is a non-exposed face.

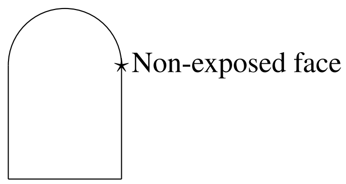


Figure 3.5: Example of a non-exposed face

Definition 3.5.3. Let Υ be a non-empty collection of convex subsets of $M_{\mathbb{R}}$. Υ is called a *convex subdivision* if the following conditions hold:

(1) Every face of an element of Υ is also in Υ.

(2) Every two elements of Υ are either disjoint or they intersect in a common face.

If only (2) is satisfied, then we call Υ a *convex decomposition*. Let Υ be a convex subdivision or decomposition of $M_{\mathbb{R}}$. The *support* of Υ is the set $|\Upsilon| := \bigcup_{C \in \Upsilon} C$. We say Υ is *complete* if its support is the whole of $M_{\mathbb{R}}$. For a given subset $E \subseteq M_{\mathbb{R}}$, if $|\Upsilon| = E$, we say Υ is a convex subdivision or decomposition of E.

Example 3.5.4. The set of all faces of a convex set K is a convex subdivision of K. The set of all exposed faces of a convex set K is a convex decomposition of K.

A concave function is said to be *closed* if it is upper semicontinuous. This includes the case of continuous, concave functions defined on closed convex sets. The *support function* of a (not necessarily bounded) convex set K is the function

$$h_K \colon N_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}} \, (= \mathbb{R} \cup \{-\infty\})$$

given by the assignment

$$v \longmapsto \inf_{m \in K} \langle m, v \rangle.$$

It is a closed, concave, conical function.

Also, recall that the *Legendre–Fenchel dual* of a concave function $f \colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$ is the closed, concave function

$$f^{\vee} \colon M_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}},$$

defined by

$$m \longmapsto \inf_{v \in N_{\mathbb{R}}} \left(\langle m, v \rangle - f(v) \right),$$

whose domain is the so called *stability set* of f, which is denoted by Δ_f.

Moreover, the *indicator function* of a convex set $K \subseteq M_{\mathbb{R}}$ is the concave function

$$\iota_K \colon M_{\mathbb{R}} \longrightarrow \underline{\mathbb{R}}$$

defined by

$$\iota_K(m) = \begin{cases} 0 & \text{if } m \in K, \\ -\infty & \text{if } m \notin K. \end{cases}$$

The following useful Remark can be found in [BPS14, Section 2.1].

Remark 3.5.5. Let $K \subseteq M_{\mathbb{R}}$ be a closed convex set and let $\iota_K \colon M_{\mathbb{R}} \to \underline{\mathbb{R}}$ be its indicator function. Then we have that $h_K = \iota_K^{\vee}$ and $h_K^{\vee} = \iota_K$. Hence, the Legendre–Fenchel duality gives a bijective correspondence between indicator functions of closed convex sets in $M_{\mathbb{R}}$ and closed, concave, conical functions on $N_{\mathbb{R}}$.

Definition 3.5.6. Let f be a concave function on $N_{\mathbb{R}}$. The *sup-differential* $\partial f(u)$ of f at $u \in N_{\mathbb{R}}$ is defined by

$$\partial f(u) := \left\{ m \in M_{\mathbb{R}} \,\middle|\, \langle m, u - v \rangle \geq f(u) - f(v), \forall v \in N_{\mathbb{R}} \right\},$$

if $f(u) \neq -\infty$, and $\emptyset$ if $f(u) = -\infty$.

This is a generalization to the non-smooth setting of the *gradient* of a smooth function at a point. Note that in general, the sup-differential may contain more than one point. Note also that if we fix an identification $N_{\mathbb{R}} \simeq \mathbb{R}^n$, then $\partial h_K|_{\mathbb{S}^{n-1}}$ is the same as the map g_K defined in the previous section.

Definition 3.5.7. We say that f is *sup-differentiable* at a point $u \in N_{\mathbb{R}}$ if $\partial f(u) \neq \emptyset$. The *effective domain* of f is the set of points where f is sup-differentiable. We denote it by $\mathrm{dom}(\partial f)$. For a subset $V \subseteq N_{\mathbb{R}}$, the set $\partial f(V)$ is defined by

$$\partial f(V) := \bigcup_{u \in V} \partial f(u).$$

In particular, the *image* of ∂f is given by $\mathrm{Im}(\partial f) = \partial f(N_{\mathbb{R}})$.

The following propositions can be found in [Roc72, Section 30].

Proposition 3.5.8. *The sup-differential $\partial f(u)$ is a closed, convex set for all $u \in \mathrm{dom}(\partial f)$. It is bounded if and only if $u \in \mathrm{relint}(\mathrm{dom}(f))$. Moreover, the effective domain of f is close to being convex, in the sense that*

$$\mathrm{relint}(\mathrm{dom}(f)) \subseteq \mathrm{dom}(\partial f) \subseteq \mathrm{dom}(f).$$

In particular, if $\mathrm{dom}(f) = N_{\mathbb{R}}$, we have $\mathrm{dom}(\partial f) = N_{\mathbb{R}}$.

Proposition 3.5.9. *If f is closed, then we have that $\mathrm{Im}(\partial f) = \mathrm{dom}(\partial f^{\vee})$. Hence, the image of the sup-differential is close to being convex, in the sense that*

$$\mathrm{relint}(\Delta_f) \subseteq \mathrm{Im}(\partial f) \subseteq \Delta_f.$$

Definition 3.5.10. Let f be a closed, concave function on $N_{\mathbb{R}}$. We denote by $\Upsilon(f)$ the collection of all sets of the form

$$C_m := \partial f^{\vee}(m) \subseteq \mathcal{P}(N_{\mathbb{R}}),$$

for $m \in \operatorname{dom}(f^{\vee}) \subseteq M_{\mathbb{R}}$.

The following is [BPS14, Proposition 2.2.8].

Proposition 3.5.11. *Let f be a closed, concave function on $N_{\mathbb{R}}$. Then $\Upsilon(f)$ is a convex decomposition of* $\operatorname{dom}(\partial f)$. *In particular, if* $\operatorname{dom}(f) = N_{\mathbb{R}}$, *then $\Upsilon(f)$ is complete.*

Definition 3.5.12. Let f be a closed, concave function on $N_{\mathbb{R}}$. The *Legendre–Fenchel* correspondence of f

$$\mathcal{L}f \colon \Upsilon(f) \longrightarrow \Upsilon(f^{\vee})$$

is given by the assignment

$$C \longmapsto \bigcap_{u \in C} \partial f(u) \quad (= \partial f(u_0),\ \text{for any } u_0 \in \operatorname{relint}(C)).$$

Definition 3.5.13. Let V, V^* be subsets of $N_{\mathbb{R}}$ and of $M_{\mathbb{R}}$, respectively. Moreover, let Υ, Υ^* be convex decompositions of V and V^*, respectively. We say that Υ and Υ^* are *dual* convex decompositions if there exists a bijective map

$$\Upsilon \longrightarrow \Upsilon^*$$

given by the assignment

$$C \longmapsto C^*$$

and satisfying the following properties:

(1) For every C, D in Υ we have that $C \subseteq D$ if and only if $C^* \supseteq D^*$.

(2) For every C in Υ, the sets C and C^* are contained in orthogonal affine spaces of $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively.

The following theorem is taken from [BPS14, Theorem 2.2.12].

Theorem 3.5.14. *Let f be a closed, concave function. Then $\mathcal{L}f$ gives a duality between $\Upsilon(f)$ and $\Upsilon(f^{\vee})$ with inverse given by $(\mathcal{L}f)^{-1} = \mathcal{L}f^{\vee}$.*

We make the following remark which can be found in [HUL01, Proposition 2.1.5].

Remark 3.5.15. Consider a closed convex set $K \subseteq M_{\mathbb{R}}$. Let h_K be the corresponding closed, concave, conical support function and let $C \in \Upsilon(h_K)$. Then, for any $u \in \operatorname{relint}(C)$ we have that $\partial h_K(u) \in \Upsilon(h_K^{\vee})$ is an exposed face of $\Delta_{h_K} = K$. Conversely, every exposed face F of K can be obtained as $\partial h_K(u)$ for some $u \in N_{\mathbb{R}}$. Explicitly, consider $m \in \operatorname{relint}(F)$. Then we may take any $u \in \operatorname{relint}(\partial h_K^{\vee}(m)) = \operatorname{relint}(\partial \iota_K(m))$. In particular, if K is bounded, we get a duality between the set of exposed faces of K and a convex decomposition of $N_{\mathbb{R}}$.

Example 3.5.16. Let notations be as in Remark 3.5.15 and assume that $K = P$ is a polytope. Then the Legendre–Fenchel duality gives back the classical duality between the faces of a polytope and the cones of its normal fan Σ_P.

If K is not polyhedral, our convex decompositions will not be finite, as can be seen in Figure 3.6. Here we have

$$h_K(a,b) = \begin{cases} \frac{ab}{a+b}, & \text{if } a,b \in \mathbb{R}_{\geq 0}, \\ \min\{0,a,b\}, & \text{otherwise.} \end{cases}$$

Note that the convex decomposition of $N_{\mathbb{R}} \simeq \mathbb{R}^2$ gives us a foliation of the positive quadrant by rays.

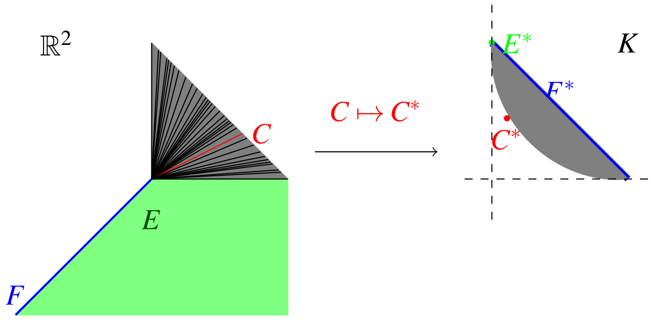


Figure 3.6: Legendre–Fenchel correspondence in the non-polyhedral case

Canonical decomposition of a difference of convex sets

We give a canonical decomposition of the complement of a full-dimensional convex set contained in another convex set.
Let $K_1 \subseteq K_2$ be two n-dimensional convex sets in $M_{\mathbb{R}}$ and let $h_{K_1}, h_{K_2} : N_{\mathbb{R}} \to \mathbb{R}$ be the corresponding concave conical support functions.

Definition 3.5.17. We define two complete convex decompositions Σ_{K_1} and Σ_{K_2} of $N_{\mathbb{R}}$ by setting

$$\Sigma_{K_i} := \Upsilon(h_{K_i})$$

for $i = 1, 2$.

Note that the elements in Σ_{K_i} for $i = 1, 2$ are cones. This follows from the fact that the convex set C_m corresponding to an $m \in \operatorname{relint}(K_i)$ is $\{0\}$. Hence, we will call Σ_{K_i} a fan, eventhough it may not be finite nor rational.

It follows from Remark 3.5.15 that the Legendre–Fenchel duality gives an order-reversing, bijective correspondence between cones in Σ_{K_i} and the set of exposed faces of K_i for $i = 1, 2$. For $F \leq K_i$ an exposed face, we will denote by σ_F the cone in Σ_{K_i} given by this correspondence.
The following is a key definition for giving the canonical decomposition of the difference $K_2 \setminus K_1$.

Definition 3.5.18. Let $F_1 \leq K_1$ and $F_2 \leq K_2$ be exposed faces. We say that F_1 *is related to* F_2 (and denote it by $F_1 \sim F_2$) if and only if

$$\operatorname{relint}(\sigma_{F_1}) \cap \operatorname{relint}(\sigma_{F_2}) \neq \emptyset$$

is satisfied.

Definition 3.5.19. Let $\Sigma \subseteq N_{\mathbb{R}}$ be a complete (not necessarily finite nor rational) fan. We say that Σ is a *difference fan* for K_1 and K_2, and denote it by $\Sigma = \Sigma_{K_2 \setminus K_1}$, if the following two conditions are satisfied:

(1) Σ is a smooth refinement of both Σ_{K_1} and Σ_{K_2}.

(2) Let $F_1 \leq K_1$ and $F_2 \leq K_2$ be exposed faces. If $F_1 \sim F_2$, then there exists a $\tau \in \Sigma(1)$ such that $\tau \in \operatorname{relint}(\sigma_{F_1}) \cap \operatorname{relint}(\sigma_{F_2})$.

Remark 3.5.20. Note that given two n-dimensional convex sets $K_1 \subseteq K_2$, we can always find a difference fan $\Sigma_{K_1 \setminus K_2}$.

The next proposition gives us the canonical decomposition of the difference $K_2 \setminus K_1$. It is one of the main results of this section.

Proposition 3.5.21. *Let $K_1 \subseteq K_2$ be two n-dimensional convex sets in $M_{\mathbb{R}}$. Then we have that*

$$\Upsilon(K_2 \setminus K_1) := \left\{ \operatorname{convhull}(F_1, F_2) \,\middle|\, F_1 \overset{\text{exposed}}{\leq} K_1, F_2 \overset{\text{exposed}}{\leq} K_2 \text{ and } F_1 \sim F_2 \right\}$$

is a convex decomposition of the difference $K_2 \setminus K_1$.

Proof. Let $F_1, F_1' \leq K_1$ and $F_2, F_2' \leq K_2$ be exposed faces such that $F_1 \sim F_1'$ and $F_2 \sim F_2'$. Now, since $F_1 \sim F_2$ we may fix a $v \in N_{\mathbb{R}}$ such that

$$F_1 = \left\{ m \in K_1 \,\middle|\, \langle v, m \rangle = \min_{m' \in K_1} \langle v, m' \rangle \right\} \quad \text{and} \quad F_2 = \left\{ m \in K_2 \,\middle|\, \langle v, m \rangle = \min_{m' \in K_2} \langle v, m' \rangle \right\}.$$

Analogously, we may fix $v' \in N_{\mathbb{R}}$ defining the faces $F_1' \sim F_2'$. Note that related faces live in parallel hyperplanes.

Now, let us show that $\operatorname{convhull}(F_1, F_2) \subseteq K_2 \setminus K_1$. Let $m \in \operatorname{convhull}(F_1, F_2)$. The fact that $m \in K_2$ is clear. Now, let λ_1, λ_2 be non-negative real numbers satisfying $\lambda_1 + \lambda_2 = 1$ and such that

$$m = \lambda_1 m_1 + \lambda_2 m_2$$

for $m_1 \in F_1$ and $m_2 \in F_2$. Since $m_1 \in F_1$, $m_2 \in F_2$ and $K_1 \subseteq K_2$, we have

$$\langle v, m_1 \rangle = \min_{m' \in K_1} \langle v, m' \rangle \geq \min_{m' \in K_2} \langle v, m' \rangle = \langle v, m_2 \rangle.$$

Hence, we obtain

$$\langle v, m \rangle = \lambda_1 \langle v, m_1 \rangle + \lambda_2 \langle v, m_2 \rangle \leq \lambda_1 \langle v, m_1 \rangle + \lambda_2 \langle v, m_1 \rangle = \langle v, m_1 \rangle,$$

which implies that

$$\operatorname{convhull}(F_1, F_2) \cap K_1 = F_1, \tag{3.10}$$

in particular $\operatorname{convhull}(F_1, F_2) \subseteq K_2 \setminus K_1$.

Now, let $m' \in K_2 \setminus K_1$. Let $F_2 \leq K_2$ be the smallest exposed face such that $m' \in F_2$. Consider a $\tau \in \operatorname{relint}(\sigma_{F_2})$ and let $F_1 \leq K_1$ be the unique exposed face such that $\tau \in \operatorname{relint}(\sigma_{F_1})$. Then $m' \in \operatorname{convhull}(F_1, F_2)$.

Hence, we have shown that

$$K_2 \setminus K_1 = \left\{ \mathrm{convhull}\,(F_1, F_2) \;\middle|\; F_1 \overset{\text{exposed}}{\leq} K_1,\, F_2 \overset{\text{exposed}}{\leq} K_2 \text{ and } F_1 \sim F_2 \right\}.$$

It remains to show that this is indeed a convex decomposition, i.e. that

$$\mathrm{convhull}\,(F_1, F_2) \cap \mathrm{convhull}\left(F_1', F_2'\right) \tag{3.11}$$

is either empty or a face of both. If the intersection is empty, then we are done. Hence, assume that $\mathrm{convhull}\,(F_1, F_2) \cap \mathrm{convhull}\,(F_1', F_2') \neq \emptyset$. By (3.10), the case in which we have that $\mathrm{convhull}(F_1, F_2) \subseteq \mathrm{convhull}(F_1', F_2')$ or in which $\mathrm{convhull}(F_1', F_2') \subseteq \mathrm{convhull}(F_1, F_2)$ is also clear. Hence, assume that

$$\mathrm{convhull}\,(F_1, F_2) \setminus \mathrm{convhull}\left(F_1', F_2'\right) \neq \emptyset$$

and

$$\mathrm{convhull}\left(F_1', F_2'\right) \setminus \mathrm{convhull}\,(F_1, F_2) \neq \emptyset.$$

Let H be a hyperplane separating F_1 and F_1'. Then, since F_1 is parallel to F_2 and F_1' is parallel to F_2', we can choose H to be a separating hyperplane of F_2 and F_2' as well. The existence of this separating hyperplane implies that

$$\mathrm{convhull}\,(F_1, F_2) \cap \mathrm{convhull}\left(F_1', F_2'\right) = \mathrm{convhull}\left(F_1 \cap F_1', F_2 \cap F_2'\right)$$

which proves that the intersection in (3.11) is a face of both. This concludes the proof of the proposition. ■

Remark 3.5.22. As we mentioned at the beginning of this chapter, in the polyhedral case, the above canonical decomposition gives a polyhedral subdivision of the complement of two polytopes, one contained in the other. This subdivision appears in the literature (e.g. in [GP88]) although it is constructed using the so called pushing method. We haven't found in the literature the method we used in Proposition 3.5.21 nor have we found such a canonical decomposition in the non-polyhedral case.

Let $K_1 \subseteq K_2$ be full-dimensional convex sets in $M_{\mathbb{R}}$.

Definition 3.5.23. Let $F \leq K_2$ be an exposed face. To F we associate the *correction set*

$$K_F := \bigcup_{\tau \in \mathrm{relint}(\sigma_F)} \mathrm{convhull}\,(F, F_{1,\tau}),$$

where for $\tau \in \mathrm{relint}\,(\sigma_F)$, the face $F_{1,\tau}$ is the unique exposed face of K_1 such that $\tau \in \mathrm{relint}\left(\sigma_{F_{1,\tau}}\right)$. The associated *correction term* c_F is defined as $n!$ times the volume of K_F, i.e.

$$c_F := n!\,\mathrm{vol}\,(K_F).$$

Remark 3.5.24. Note that by Proposition 3.5.21 we have that

$$K_2 \setminus K_1 = \bigcup_{F \overset{\text{exposed}}{\leq} K_2} K_F$$

is a convex decomposition and hence

$$n!\,\mathrm{vol}\,(K_2 \setminus K_1) = \sum_{F \overset{\text{exposed}}{\leq} K_2} c_F.$$

Let’s look at a simple 2-dimensional polyhedral example.

Example 3.5.25. Consider the simplex K_1 contained in the square K_2 as in the Figure 3.7. Here, the different colors show the correction sets associated to the faces of K_2. Figure 3.8 shows the dual picture with the fans $\Sigma_{K_2}, \Sigma_{K_2\setminus K_1}, \Sigma_{K_1}$.

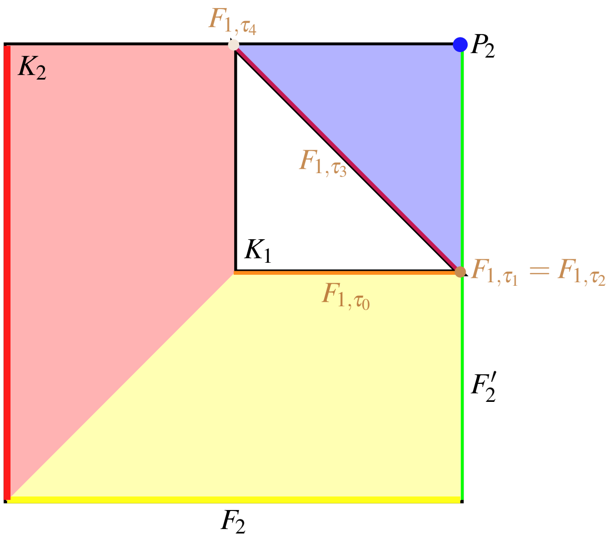


Figure 3.7: Canonical decomposition of the complement of the simplex contained in the square

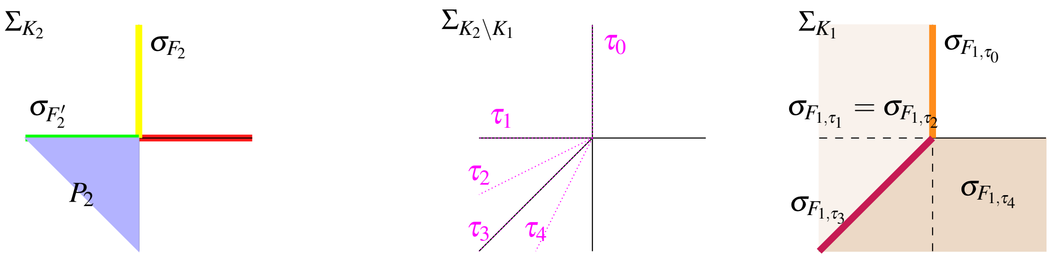


Figure 3.8: Difference fan of the simplex contained in the square

Volumes and intersection numbers

We relate the correction terms of Definition 3.5.23 with tropical intersection numbers of continuous tropically nef b-divisors in the case that we start with two full-dimensional convex sets $K_1 \subseteq K_2$ whose support function is rational and such that K_2 is a polytope.

Note that two related exposed faces $F_1 \sim F_2$ with $F_1 \leq K_1$ and $F_2 \leq K_2$ are contained in parallel hyperplane sections (defined by the v given in the proof of Proposition 3.5.21). The following is a key Lemma.

Lemma 3.5.26. *Let $F_1, F_2 \subseteq \mathbb{R}^{d+1}$ be polytopes. Here, $d = \max\{\dim(F_1), \dim(F_2)\}$. Assume that*

$F_1 \subseteq \{x_{d+1} = 0\}$ *and that* $F_2 \subseteq \{x_{d+1} = 1\}$. *Then the volume of the convex hull of* F_1, F_2 *is given by*

$$\operatorname{vol}(\operatorname{convhull}(F_1,F_2)) = \frac{1}{d+1}\sum_{i=0}^{d} \operatorname{MV}\left(\underbrace{F_1,\ldots,F_1}_{i\text{-times}},\underbrace{F_2,\ldots,F_2}_{(d-i)\text{-times}}\right).$$

Proof. We start with the following three claims:
Claim 1: Let λ be any real number between 0 and 1. Then the slice of the convex hull of F_1 and F_2 at $x_{d+1} = \lambda$ is given by

$$\operatorname{convhull}(F_1,F_2) \cap \{x_{d+1} = \lambda\} = \lambda F_1 + (1-\lambda)F_2,$$

where the sum in the right hand side is the Minkowski sum of convex sets.
Claim 2: Let λ be any real number between 0 and 1. Then it follows that the volume of the slice $\lambda F_1 + (1-\lambda)F_2 \subseteq \operatorname{convhull}(F_1,F_2)$ is given by

$$\operatorname{vol}(\lambda F_1 + (1-\lambda)F_2) = \sum_{i=0}^{d}\binom{d}{i}\lambda^i(1-\lambda)^{d-i}\operatorname{MV}\left(\underbrace{F_1,\ldots,F_1}_{i\text{-times}},\underbrace{F_2,\ldots,F_2}_{(d-i)\text{-times}}\right).$$

Claim 3: Let λ be as before and let ℓ, k be two non-negative integers with $k \leq \ell$. We define the number $\mathrm{I}(\ell,k)$ by

$$\mathrm{I}(\ell,k) := \int_0^1 \lambda^k (1-\lambda)^{\ell-k}\,\mathrm{d}\lambda.$$

Then the formula

$$\mathrm{I}(\ell,k) = \left((\ell+1)\binom{\ell}{k}\right)^{-1}$$

holds true.

Now, Claim 1 is clear and Claim 2 is a standard result in convex geometry. We proceed to give a proof of Claim 3: integrating by parts, we get

$$\begin{aligned}
\mathrm{I}(\ell,k) &= \int_0^1 \lambda^k(1-\lambda)^{\ell-k}\,\mathrm{d}\lambda \\
&= \frac{\lambda^{k+1}(1-\lambda)^k}{k+1}\bigg|_0^1 + \int_0^1 \frac{\lambda^{k+1}}{k+1}(\ell-k)(1-\lambda)^{\ell-k+1}\,\mathrm{d}\lambda \\
&= \frac{\ell-k}{k+1}\,\mathrm{I}(\ell,k+1).
\end{aligned}$$

Moreover the values for $k = \ell$ and for $k = 0$ are given by

$$\begin{aligned}
\mathrm{I}(\ell,\ell) &= \int_0^1 \lambda^\ell\,\mathrm{d}\lambda = \frac{\lambda^{\ell+1}}{\ell+1}\bigg|_0^1 = \frac{1}{\ell+1}, \\
\mathrm{I}(\ell,0) &= \int_0^1 (1-\lambda)^\ell\,\mathrm{d}\lambda = \frac{-(1-\lambda)^{\ell+1}}{\ell+1}\bigg|_0^1 = \frac{1}{\ell+1}.
\end{aligned}$$

Hence, we get

$$
\begin{aligned}
\mathrm{I}(\ell,\ell-1) &= \frac{1}{\ell}\cdot\frac{1}{\ell+1},\\
\mathrm{I}(\ell,\ell-2) &= \frac{2}{\ell-1}\cdot\frac{1}{\ell}\cdot\frac{1}{\ell+1},\\
&\dots\\
\mathrm{I}(\ell,k) &= \left((\ell+1)\binom{\ell}{k}\right)^{-1},
\end{aligned}
$$

as we wanted to show.

Finally, note that Claim 1, Claim 2 and Claim 3 imply that

$$
\begin{aligned}
\operatorname{vol}(\operatorname{convhull}(F_1,F_2)) &= \int_0^1 \operatorname{vol}(\lambda F_1+(1-\lambda)F_2)\,\mathrm{d}\lambda\\
&= \int_0^1 \sum_{i=0}^{d}\binom{d}{i}\lambda^i(1-\lambda)^{d-i}\operatorname{MV}\left(\underbrace{F_1,\dots,F_1}_{i\text{-times}},\underbrace{F_2,\dots,F_2}_{(d-i)\text{-times}}\right)\mathrm{d}\lambda\\
&= \sum_{i=0}^{d}\binom{d}{i}\mathrm{I}(d,i)\operatorname{MV}(F_1,\dots,F_1,F_2,\dots,F_2)\\
&= \frac{1}{d+1}\sum_{i=0}^{d}\operatorname{MV}(F_1,\dots,F_1,F_2,\dots,F_2),
\end{aligned}
$$

concluding the proof of the lemma. ∎

Let $K_1 \subseteq K_2$ be two full-dimensional convex sets with corresponding support functions ϕ_1,ϕ_2. Moreover, let $\Sigma = \Sigma_{K_2\setminus K_1} \subseteq N_{\mathbb{R}}$ be a difference fan. We have the following theorem.

Theorem 3.5.27. *Let notations be as above and assume that K_2 is a polytope. Then the functions ϕ_1 and ϕ_2 can be seen as continuous tropically nef b-divisors on the normal fan of K_2, denoted by Σ_{K_2}. We can express the difference $\phi_2^n-\phi_1^n$ of degrees of the continuous tropically nef b-divisors ϕ_1 and ϕ_2 as a finite sum of correction terms*

$$
\phi_2^n-\phi_1^n = \sum_{F \overset{\text{exposed}}{\leq} K_2} c_F,
$$

where the correction terms c_F are related to tropical intersection numbers by

$$
\begin{aligned}
c_F &= \sum_{i=0}^{n-1}(n-1)!\int_{\operatorname{relint}(\sigma_F)\cap\mathbb{S}^{n-1}}(\phi_1(u)-\phi_2(u))\,S(\underbrace{K_1,\dots,K_1}_{i\text{-times}},\underbrace{K_2,\dots,K_2}_{(n-1-i)\text{-times}},u)\\
&= \sum_{i=0}^{n-1}\int_{\operatorname{relint}(\sigma_F)\cap\mathbb{S}^{n-1}}(\phi_1(u)-\phi_2(u))\,\mu_{\underbrace{\phi_1,\dots,\phi_1}_{i\text{-times}},\underbrace{\phi_2,\dots,\phi_2}_{(n-1-i)\text{-times}}},
\end{aligned}
$$

where $S(\cdot)$ is the mixed surface area measure defined in the previous section and $\mu_{\phi_1,\dots,\phi_2,\dots}$ is the mixed limit measure. In particular, if Δ_1 is also polyhedral, we get

$$
c_F = \sum_{i=0}^{n-1}\sum_{\substack{\tau\in\operatorname{relint}(\sigma_F)\\ \tau\in\Sigma_{K_2}(1)}}(\phi_1(v_\tau)-\phi_2(v_\tau))\left((\phi_1(\tau))^{n-1-i}(\phi_2(\tau))^i\right)\cdot[\Sigma_{K_2}(\tau)],
$$

where the "$\cdot$" is the tropical intersection product and the $\phi_i(\tau)$'s are the piecewise linear support functions on the star $\Sigma_{K_2}(\tau)$ as in Theorem 3.4.22.

Proof. The first and last statement of the theorem follow from Proposition 3.5.21 and Definition 3.5.23. For the statement regarding the expression of the correction terms in terms of intersection numbers, we have

$$\begin{aligned}\phi_2^n - \phi_1^n &= (\phi_2 - \phi_1)\sum_{i=0}^{n-1} \phi_1^i \phi_2^{n-1-i} \\ &= \sum_{\sigma \in \Sigma_{K_2}(2)} \sum_{i=0}^{n-1} \int_{\mathrm{relint}(\sigma)\cap \mathbb{S}^{n-1}} (\phi_2(u) - \phi_1(u))\, \mu_{\underbrace{\phi_1,\dots,\phi_1}_{i\text{-times}},\underbrace{\phi_2,\dots,\phi_2}_{(n-1-i)\text{-times}}} \\ &= \sum_{\substack{\text{exposed} \\ F \;\leq\; K_2}} \sum_{i=0}^{n-1} (n-1)! \int_{\mathrm{relint}(\sigma_F)\cap \mathbb{S}^{n-1}} (\phi_1(u) - \phi_2(u))\, S(\underbrace{K_1,\dots,K_1}_{i\text{-times}}, \underbrace{K_2,\dots,K_2}_{(n-1-i)\text{-times}}, u),\end{aligned}$$

concluding the proof of the theorem. ■

Example 3.5.28. Let us take our usual 2-dimensional example: consider the fan of $\mathbb{P}^2$ and the continuous tropically nef b-divisors ϕ_1 and ϕ_2 given by the concave functions $\phi_1, \phi_2 \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi_1(a,b) = \begin{cases} \frac{ab}{a+b}\,, & a,b \in \mathbb{R}_{\geq 0}, \\ \min\{0,a,b\}\,, & \text{otherwise.} \end{cases}$$

and

$$\phi_2(a,b) = \min\{0,a,b\}$$

and consider the corresponding convex sets $K_1 \subseteq K_2$. The only face of the simplex K_2 whose associated correction term is non-zero is the vertex F_0 in Figure 3.9.

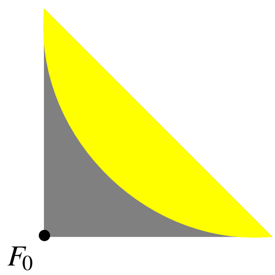


Figure 3.9: Convex sets $K_1 \subseteq K_2$

On the one hand, we can calculate the difference $\phi_2^2 - \phi_1^2$ as the difference of volumes of convex sets

$$c_{F_0} = \text{ full correction term } = \phi_2^2 - \phi_1^2 = 2\,\mathrm{vol}(\Delta_2) - 2\,\mathrm{vol}(\Delta_1) = 1 - \frac{2}{3} = \frac{1}{3}.$$

On the other hand, Theorem 3.5.27 tells us that we can compute the correction term c_{F_0} by

$$c_{F_0} = \int_0^{\pi/2} \phi_1(\theta) S_1(K_1,\theta) = \frac{1}{3} = \int_0^{\pi/2} \phi_1(\theta) s_1(K_1,\theta) = \frac{1}{3},$$

where the last two equalities follow from the computations done in Example 3.4.9 and Example 3.4.19.

In Appendix C we give a 3-dimensional example where we compute correction terms in the polyhedral case.

Chapter 4

Intersection theory of b-divisors on toroidal varieties

The aim of this chapter is to generalize the intersection theory of nef toric b-divisors on smooth toric varieties to more general nef toroidal b-divisors on smooth toroidal varieties. The key ingredient to do this is the tropical measure μ_ϕ induced by a continuous tropically nef b-divisor ϕ on a weakly embedded rational conical polyhedral complex as defined in the previous chapter. As before, we will refer to a rational conical polyhedral complex just as a conical complex.

We start by recalling standard notation and basic facts about toroidal embeddings and their associated conical complexes. A more detailed introduction to this theory can be found in [KKMSD73] or in [AMRT10]. We also describe their corresponding weak embeddings for which our main reference is [Gro15]. Moreover, we show how one can modify the toroidal structure of a given toroidal embedding in such a way that the corresponding conical complex becomes combinatorially principal (Definition 3.2.8). We then define toroidal b-divisors on a smooth toroidal embedding as an inverse limit of toroidal divisors indexed over all smooth subdivisions of the weakly embedded, smooth, conical complex attached to the toroidal embedding. Generalizing the toric setting, we show that a nef toroidal b-divisor on a smooth toroidal embedding can be seen as a continuous function on the support of the weakly embedded conical complex whose restriction to each cone is concave. Thus, we may view nef toroidal b-divisors as continuous tropically nef b-divisors on the conical complex (Definition 3.3.4). Finally, we define the mixed degree of a collection of toroidal b-divisors as a limit of top intersection numbers of toroidal divisors. The main result of this chapter is that if the toroidal b-divisors we start with are nef, then their mixed degree exists and is finite. This is proven as follows: we make a slight detour into tropical geometry and we recall the definition of the tropicalization map which associates a tropical cycle to an algebraic cycle. This allows us to compute the mixed degree of nef toroidal b-divisors as the mixed degree of continuous tropically nef b-divisors on a weakly embedded conical complex as was done in Chapter 3 by using the combinatorially principal conical complex induced by the modified toroidal structure. As a corollary, we give a formula for the difference of the self intersection number of any (not necessarily toroidal) b-divisor on a smooth toroidal embedding and its incarnation to a smooth toroidal model.

The concept of a toroidal b-divisor given in this chapter seems to be new as well as the theory of toroidal b-divisors developed here.

4.1 Toroidal embeddings and rational conical polyhedral complexes

Throughout this chapter, k will denote an algebraically closed field of characteristic 0. In this section, we recall the weakly embedded conical complex associated to a toroidal embedding. We also show that by adding boundary components one can modify the toroidal structure of a toroidal embedding in such a way that the corresponding weakly embedded conical complex becomes combinatorially principal, abbreviated as cp (Definition 3.2.8). Then we describe the proper toroidal birational modifications of the toroidal embedding which, on the combinatorial side, correspond to projective subdivisions of the conical complex (Definition 3.1.8). We refer to [KKMSD73] and to [AMRT10] for more details about toroidal varieties and their associated conical complexes and to [Gro15] for their associated weak embeddings.

Toroidal embeddings

We give the definition of a toroidal embedding and describe its associated conical complex. We also give a natural weak embedding of this conical complex. The following definition is taken from [KKMSD73, Definition 1, pg. 54].

Definition 4.1.1. Let X be an n-dimensional normal, irreducible variety over k and let U be a smooth Zariski open subset of X. We say that $U \hookrightarrow X$ is a *toroidal embedding* if for every closed point $x \in X$ there exists an n-dimensional torus $\mathbb{T}$, an affine toric variety $X_\sigma \supseteq \mathbb{T}$, a point $x' \in X_\sigma$ and an isomorphism of k-local algebras

$$\hat{\mathcal{O}}_{X,x} \xrightarrow{\simeq} \hat{\mathcal{O}}_{X_\sigma,x'} \tag{4.1}$$

such that the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal of $X \setminus U$ corresponds under this isomorphism to the ideal in $\hat{\mathcal{O}}_{X_\sigma,x'}$ generated by the ideal of $X_\sigma \setminus \mathbb{T}$. Here, the hat "ˆ" refers to the completion of the local ring at a point. Such an isomorphism is called a *chart* at x and the pair (X_σ, x') is called a *local model* at x.

Throughout this chapter we will always assume that the irreducible components of the boundary divisor $X \setminus U$ of a toroidal embedding are normal. This is referred to as being a *toroidal embedding without self intersection*.

Remark 4.1.2. The last condition in Definition 4.1.1 implies that the isomorphism (4.1) maps the ideal of the boundary divisor $X \setminus U$ to the ideal of the *toric* boundary divisor $X_\sigma \setminus \mathbb{T}$. Hence, X has a stratification that behaves like the stratification into orbits of a toric variety and thus can be described combinatorially by a conical complex as we now describe.

Definition 4.1.3. Let $U \hookrightarrow X$ be a toroidal embedding and let $\{B_i \,|\, i \in I\}$ be the irreducible components of the boundary divisor $B = X \setminus U$. For every subset $J \subseteq I$ such that $B_J := \bigcap_{i \in J} B_i \neq \emptyset$, we define a stratum S_J for every connected component of the set $\bigcap_{i\in J} B_i \setminus \bigcup_{i \notin J} B_i$. For $J = \emptyset$, the stratum $S_\emptyset$ is defined to be the open subset $U \subseteq X$.

The following lemma is [KKMSD73, Proposition-Definition 2, pg. 57].

Lemma 4.1.4. *Let notations be as above and consider a subset $J \subseteq I$ such that $B_J := \bigcap_{i\in J} B_i \neq \emptyset$. Then the following holds true:*

(1) B_J is normal.

(2) S_J is non-singular.

Moreover, the sets S_J define a stratification of X, i.e. every point of X is in exactly one stratum and the closure of a stratum is a union of strata. Furthermore, if $x \in X$ and (X_σ, x') is a local model at x, then the closures $\overline{S}_J$ of the strata S_J such that $x \in \overline{S}_J$ correspond formally to the closure of the torus orbits in X_σ containing x'. In particular, if $x \in S_J$, then S_J corresponds formally to the torus orbit $O(x')$ itself.

The following Proposition/Definition is adapted from [KKMSD73, Definition 3, pg. 59].

Proposition/Definition 4.1.5. *With notations as in Definition 4.1.3, let S_J be a non-empty stratum of the toroidal embedding $U \hookrightarrow X$. The combinatorial open set* $\mathrm{Star}(S_J)$ *is defined by*

$$\mathrm{Star}(S_J) := \bigcup_{K \subseteq I\colon S_J \subseteq \overline{S}_K} S_K = X \setminus \bigcup_{L \subseteq I\colon \overline{S}_L \cap S_J = \emptyset} S_L,$$

where $\overline{S}_K$ and $\overline{S}_L$ denote the closures of the strata S_K and S_L, respectively. The sets M^{S_J} and $M_+^{S_J}$ are defined by

$$\begin{aligned} M^{S_J} &:= \left\{ B \in \text{Ca-Div}\left(\mathrm{Star}\left(S_J\right)\right) \,\middle|\, \mathrm{supp}(B) \subseteq \mathrm{Star}\left(S_J\right) \setminus U \right\}, \\ M_+^{S_J} &:= \left\{ B \in M^{S_J} \,\middle|\, B \textit{ effective} \right\}. \end{aligned}$$

We have that M^{S_J} is a free abelian group (a lattice) while $M_+^{S_J}$ has the structure of a sub-semigroup. For each stratum S_J we denote by N^{S_J} the dual lattice of M^{S_J} and by $\langle , \rangle_{S^J}$ the induced pairing. Finally, the set σ^{S_J} is defined by

$$\sigma^{S_J} := \left\{ v \in N_{\mathbb{R}}^{S_J} \,\middle|\, \langle B, v \rangle_{S^J} \geq 0, \forall B \in M_+^{S_J} \right\} \subseteq N_{\mathbb{R}}^{S_J},$$

where $N^{S_J} = \left(M^{S_J}\right)^\vee$ denotes the dual lattice of M^{S_J}. The set σ^{S_J} is a strongly convex rational polyhedral cone.

The idea behind Proposition/Definition 4.1.5 is that given a stratum S, we can produce a cone σ^S in the finite-dimensional real vector space $N_{\mathbb{R}}^S$ which comes equipped with a canonical lattice N^S. These cones can be glued together into a conical complex (Definition 3.1.1). The following result can be found in [KKMSD73, Chapter II].

Proposition 4.1.6. *If $U \hookrightarrow X$ is a toroidal embedding, then the triple*

$$\Pi_X = \left(\left| \bigsqcup_{S_J \,\text{stratum}} \mathrm{relint}\left(\sigma^{S_J}\right) \right|, \{\sigma^{S_J}\}_{S_J \,\text{stratum}}, \{M^{S_J}\}_{S_J \,\text{stratum}} \right)$$

is a conical complex.

The collection of lattices $\{M^{S_J}\}$ in the above proposition gives the *integral structure* of the toroidal embedding. We will usually omit the index S^J from the pairing $\langle , \rangle_{S^J}$.

Remark 4.1.7. The conical complex Π_X associated to a toroidal embedding $U \hookrightarrow X$ is not necessarily connected. However, for the rest of this chapter we will assume it to be connected. This is not a strong assumption, we do it to avoid more tedious notation and to be able to make a direct connection with Chapter 3.

The following lemma ([KKMSD73, Chapter II, Corollary 1]) will be useful later.

Lemma 4.1.8. *Let $U \hookrightarrow X$ be a toroidal embedding and let $x \in X$ belong to a stratum S. If (X_σ, x') is a local model at x then we have that*

$$M^S \simeq M(\mathbb{T})/\left(M(\mathbb{T}) \cap \sigma^\perp\right) \quad \textit{and} \quad \sigma^S \simeq \sigma\,,$$

where $M(\mathbb{T})$ refers to the lattice of characters of the torus $\mathbb{T} \subseteq X_\sigma$. In particular, the local model (X_σ, x') is uniquely determined by the stratum S.

Given a cone σ in Π_X, we will denote by S^σ the stratum corresponding to σ and by $\overline{S^\sigma}$ its closure in X.

Example 4.1.9. Let Σ be a fan in $N_\mathbb{R}$ for some lattice N and let $M := N^\vee$ be its dual lattice. Furthermore, let X_Σ be its associated normal toric variety with dense torus $\mathbb{T} = \operatorname{Spec}(k[M])$. Clearly, the inclusion $\mathbb{T} \hookrightarrow X_\Sigma$ defines a toroidal embedding. The components of the boundary divisor $B = X_\Sigma \setminus \mathbb{T}$ are the $\mathbb{T}$-invariant prime divisors B_τ corresponding to the rays $\tau \in \Sigma(1)$, and the strata of X are the $\mathbb{T}$-orbits $O(\sigma)$ corresponding to the cones $\sigma \in \Sigma$ (see Section 1.1). The combinatorial open sets of X_Σ are precisely its $\mathbb{T}$-invariant affine open subsets. For every cone $\sigma \in \Sigma$, we write $\sigma^\perp$ for the set defined by

$$\sigma^\perp := \{m \in M \,|\, \langle m, v\rangle = 0, \forall v \in \sigma\}.$$

Then the isomorphism

$$M/\left(M \cap \sigma^\perp\right) \simeq M^{O(\sigma)}$$

given by the assignment

$$[m] \longmapsto \operatorname{div}(\chi^m)\,,$$

where χ^m denotes the character of the torus associated to $m \in M$, induces an identification of lattices

$$N^{O(\sigma)} \simeq N_\sigma = N \cap \operatorname{Span}(\sigma)$$

and of cones

$$\sigma^{O(\sigma)} \simeq \sigma.$$

Remark 4.1.10. As in the toric case, the rays of the conical complex $\Pi_X(1)$ associated to a toroidal embedding $U \hookrightarrow X$ are in bijective correspondence with the irreducible components of the boundary divisor $B = X \setminus U$. Indeed for every irreducible component B_i, the corresponding ray in $\Pi_X(1)$, which we will denote by τ_{B_i} is the linear function $\tau_{B_i} \colon M^{S_{\{i\}}} \to \mathbb{Z}$ given by $nB_i \mapsto n$. Conversely, one can show that any ray $\tau \in \Pi_X(1)$ arises in this way (see [KKMSD73, pg. 63]). For any such ray τ, we will denote by B_τ the corresponding irreducible boundary component.

Before giving a more general class of examples of toroidal embeddings, we make some definitions.

Definition 4.1.11. Let $B \subseteq X$ be a divisor on a smooth variety X. We say that B is a *simple normal crossing divisor* (abbreviated snc) if the following three conditions hold:

(1) Every irreducible component of B is smooth.

(2) All intersections are transverse, i.e. for all $x \in X$ we can choose local coordinates $x_1, \dots, x_n$ and natural numbers $\ell_1, \dots, \ell_n$ such that $B = \left\{\prod_i x_i^{\ell_i} = 0\right\}$ in a neighborhood of x.

(3) If $\{B_i\}_{i \in I}$ are the irreducible components of B, then $B_J = \bigcap_{j \in J} B_j$ is either irreducible or $\emptyset$ for all subsets $J \subseteq I$.

A *normal crossing divisor* (abbreviated nc) is a simple normal crossing divisor without conditions (1) and (3).

Definition 4.1.12. Let (X,B) be a pair consisting of an algebraic variety X and a snc divisor B on X. Let $\{B_i\}_{i\in I}$ be the irreducible components of B. The *Clemens complex* associated to this pair is defined as the simplicial subcomplex Δ_X of the standard simplex $\Delta^{\#I}$ characterized by the property that for all non-empty subsets $J\subseteq I$, the standard simplex $\Delta^{\#J}$ is a face of Δ_X if and only if $B_J\neq\emptyset$.

We can now give a large class of examples of toroidal varieties.

Example 4.1.13. Let (X,B) be a pair consisting of a smooth projective variety X together with a nc divisor $B\subseteq X$. We denote by $\{B_i\}_{i\in I}$ the irreducible components of B. Set $U:=X\setminus B$. Then $U\hookrightarrow X$ is a toroidal embedding. If B is moreover snc, then the conical complex associated to the toroidal embedding $U\hookrightarrow X$ is smooth and can be identified with the conical complex defined by taking the cones over the faces of the Clemens complex of the pair (X,B).

Remark 4.1.14. It follows from the definition of a toroidal embedding $U\hookrightarrow X$ that the boundary $X\setminus U$ is a divisor, however, it may not be snc nor nc. Nevertheless, by Hironaka's resolution of singularities [Hir64], we can always find an allowable modification of $U\hookrightarrow X'$ (Definition 4.1.22) such that the boundary divisor $X'\setminus U$ is snc.

The next example appears when one studies toroidal compactifications of the moduli space of principally polarized abelian varieties.

Example 4.1.15. Let $\mathcal{X},S$ be algebraic varieties over k and suppose that $\pi\colon\mathcal{X}\to S$ is a torus bundle with fiber $\mathbb{T}\simeq(k^\times)^r$. Let $N=\mathrm{Hom}(k^\times,\mathbb{T})$ be the lattice of 1-parameter subgroups of the torus $\mathbb{T}$ and let Σ be any fan contained in $N_{\mathbb{R}}$. Then we may form the bundle $\mathcal{X}_\Sigma:=\mathcal{X}\times_{\mathbb{T}}X_\Sigma$ associated to Σ. If $\bar{\pi}\colon\mathcal{X}_\Sigma\to S$ denotes the projection morphism, then there is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\iota"] \arrow[d, "\pi"'] & \mathcal{X}_\Sigma \arrow[d, "\bar{\pi}"] \\ S \arrow[r, "="] & S \end{tikzcd}$$

where ι restricts in each fiber to the inclusion $\mathbb{T}\hookrightarrow X_\Sigma$. Moreover, $\mathcal{X}_\Sigma\setminus\mathcal{X}\hookrightarrow\mathcal{X}_\Sigma$ defines a toroidal embedding (see [HKW93, Remark 3.52]).

The following definition gives a natural weak embedding of the conical complex coming from a toroidal embedding. It is taken from [Gro15, Section 2.3].

Definition 4.1.16. Let $U\hookrightarrow X$ be a toroidal embedding. The set M^{Π_X} is defined to be the lattice of invertible regular functions on the open set U modulo constants, i.e.

$$M^{\Pi_X}:=\Gamma\left(U,\mathcal{O}_X^\times\right)/k^\times,$$

and $N^{\Pi_X}:=\left(M^{\Pi_X}\right)^\vee$ denotes its dual lattice; the seemingly complicated notation will become clear in a moment. For every stratum S of X we have a morphism of lattices $M^{\Pi_X}\to M^S$ given by

$$f\longmapsto \mathrm{div}(f)|_{\mathrm{Star}(S)}.$$

Dualizing, we get a linear map $\sigma^S \to N_{\mathbb{R}}^{\Pi_X}$. These maps glue to give a continuous function

$$\iota_{\Pi_X}\colon\ |\Pi_X| \longrightarrow N_{\mathbb{R}}^{\Pi_X},$$

which is integral linear on the cones of Π_X, i.e. we obtain a weakly embedded conical complex associated to X which we also denote by Π_X (Definition 3.1.13).

The following two examples are taken from [Gro15, Example 2.2].

Example 4.1.17. (1) Consider the toric setting $X = X_\Sigma$ from Example 4.1.9. Here, we have the lattice $M^{\Pi_{X_\Sigma}} = \Gamma\left(\mathbb{T}, \mathcal{O}_{X_\Sigma}^\times\right)/k^\times$, which we can identify with M via the isomorphism $M \simeq M^{\Pi_{X_\Sigma}}$ given by the assignment

$$m \longmapsto \chi^m.$$

We see that the image of $\sigma^{O(\sigma)}$ in $N_{\mathbb{R}}$ under the weak embedding $\iota_{\Pi_{X_\Sigma}}$ is precisely σ. Hence, Π_{X_Σ} is a weakly embedded conical complex, naturally isomorphic to Σ. This is an example of a weakly embedded conical complex which is cp (Definition 3.2.8).
(2) For a non-toric example, consider $X = \mathbb{P}^2$ with affine coordinates x_1, x_2, x_3 but with open part U given by

$$U = X \setminus (H_1 \cup H_2),$$

where H_i is the hyperplane given by $\{x_i = 0\}$. This is a toroidal embedding with snc boundary divisor and we see that the conical complex Π_X is naturally identified with the non-negative orthant $\mathbb{R}^2_{\geq 0}$, whose rays $\mathbb{R}_{\geq 0}(1,0)$ and $\mathbb{R}_{\geq 0}(0,1)$ correspond to the divisors H_1 and H_2, respectively. The lattice M^{Π_X} is generated by x_1/x_2, and using that generator to identify M^{Π_X} with $\mathbb{Z}$, we see that the weak embedding ι_{Π_X} sends $(1,0)$ to 1 and $(0,1)$ to -1. Note that in this case Π_X is not cp as the cone $\mathbb{R}^2_{\geq 0}$ gets mapped to 0.

The following key proposition says that given a toroidal embedding with nc boundary divisor, we can always modify the toroidal structure in such a way that the associated weakly embedded conical complex becomes cp. It will be important for us later on.

Proposition 4.1.18. *Let $U \hookrightarrow X$ be a toroidal embedding such that the boundary divisor $B = X \setminus U$ is nc and let $\Pi = \Pi_X$ be its associated weakly embedded conical complex. Then there exists a nc divisor B' with $|B| \subseteq |B'|$ such that the weakly embedded conical complex $\Pi' = \Pi'_X$ associated to the toroidal embedding $U' \hookrightarrow X$, where $U' := X \setminus B'$, is cp, i.e. the restriction of the weak embedding*

$$\iota_{\Pi'}|_{\sigma'}\colon\ |\Pi'| \longrightarrow N_{\mathbb{R}}^{\Pi'}$$

to any cone $\sigma' \in \Pi'$ is injective. Moreover, the natural projection $\Pi' \to \Pi$ induces a morphism of weakly embedded conical complexes as in Definition 3.1.15.

Proof. Recall that n is the dimension of X. First, let us suppose that $B \subseteq X$ is a prime divisor. It is a well known fact that any divisor is linearly equivalent to the difference of two very ample divisors. Then, using that the linear system of a very ample divisor is of dimension bigger than n and the moving lemma (see e.g. [Har00, Appendix A]), we can find very ample divisors $A_1, \ldots, A_{n+1}, C_1, C_2$ satisfying the following properties:

(1) $B \sim A_1 - C_1$,

(2) $C_1 \sim C_2$,

(3) $A_1 \sim A_i$, for all $i = 2, \ldots, n+1$,

(4) The A_i's, the C_i's and B all intersect each other transversally.

We now define the divisor $B' \subseteq X$ by

$$B' := B + \sum_{i=1}^{n+1} A_i + C_1 + C_2.$$

Note that the divisor B' is nc. In particular, we get a toroidal embedding $U' := X \setminus B' \hookrightarrow X$ (see Example 4.1.13). We denote by Π' the weakly embedded conical complex corresponding to it. We claim that Π' is cp. In order to prove the claim note that $\mathrm{rk}(M^{\Pi'}) \geq n+1$ since by the above choice of divisors, namely property (4), we have $(n+1)$ linearly independent rational functions $f_1, \dots, f_{n+1}$ in $\Gamma\left(U', \mathcal{O}_X^{\times}\right)$ satisfying

$$\begin{aligned} \mathrm{div}(f_1) &= B - (A_1 - C_1), \\ \mathrm{div}(f_2) &= C_1 - C_2, \\ \mathrm{div}(f_{i+1}) &= A_1 - A_i \ \text{ for } i = 2, \dots, n+1. \end{aligned}$$

We may assume that the first $n+1$ coordinates of the weak embedding $\iota_{\Pi'} \colon |\Pi'| \to N_{\mathbb{R}}^{\Pi'}$ are given by the order of vanishing of the $f_i's$ along the components of the boundary divisor. In particular, denoting by τ_D the ray in $\Pi(1)$ corresponding to a prime boundary divisor D we obtain

$$\iota_{\Pi'}(\tau_B) = (\mathrm{ord}_B(f_1), \mathrm{ord}_B(f_2), \dots, \mathrm{ord}_B(f_{n+1}), \dots) = (\underbrace{1, 0, \dots, 0}_{(n+1)\text{-times}}, \dots)$$

$$\iota_{\Pi'}(\tau_{C_1}) = (\mathrm{ord}_{C_1}(f_1), \mathrm{ord}_{C_1}(f_2), \dots, \mathrm{ord}_{C_1}(f_{n+1}), \dots) = (\underbrace{1, 1, 0, \dots, 0}_{(n+1)\text{-times}}, \dots)$$

$$\iota_{\Pi'}(\tau_{C_2}) = (\mathrm{ord}_{C_2}(f_1), \mathrm{ord}_{C_2}(f_2), \dots, \mathrm{ord}_{C_2}(f_{n+1}), \dots) = (\underbrace{0, -1, 0, \dots, 0}_{(n+1)\text{-times}}, \dots)$$

$$\iota_{\Pi'}(\tau_{A_1}) = (\mathrm{ord}_{A_1}(f_1), \mathrm{ord}_{A_1}(f_2), \dots, \mathrm{ord}_{A_1}(f_{n+1}), \dots) = (\underbrace{-1, 0, 1, 1, \dots, 1}_{(n+1)\text{-times}}, \dots)$$

$$\iota_{\Pi'}(\tau_{A_i}) = (\mathrm{ord}_{A_i}(f_1), \mathrm{ord}_{A_i}(f_2), \dots, \mathrm{ord}_{A_i}(f_{n+1}), \dots) = (\underbrace{0, 0, \dots, 0, -1, 0, \dots, 0}_{(n+1)\text{-times, } i\text{-th position}}, \dots).$$

Note that any collection of vectors of size up to n having the above form is linearly independent. Therefore, since every cone in Π' is generated by a collection of rays as above and is of dimension at most n, it follows that the restriction of $\iota_{\Pi'}$ to any cone is injective.

In the general case where B is not necessarily a prime divisor, we denote by $\{B_i\}_{i=1}^r$ its irreducible components. For every component $B_1, \dots, B_r$ we can inductively find divisors $B'_1, \dots, B'_r$ satisfying the following properties:

(1) For all $i = 1, \dots, r$, $B'_i \sim B_i$.

(2) For all $i = 1, \dots, r$, B'_i intersects transversally all components of B and of B'_j for all $j \leq i$.

We define the divisor $B' := \sum_{i=1}^r B'_i$ and the open set $U' := X \setminus B'$. Note that for every component B_i, for $i \in \{1, \dots, r\}$, we have found a rational function g_i in $\Gamma\left(X \setminus B', \mathcal{O}_X^{\times}\right)$ such that the order of vanishing of g_i along B_j is δ_{ij} for every $j \in \{1, \dots, r\}$. This, together with the prime divisor case, implies that the weakly embedded conical complex associated to the toroidal embedding $U' \hookrightarrow X$ is cp.

To show the second part of the statement of the proposition, we have to show that the diagram

$$
\begin{tikzcd}
|\Pi'| \arrow[r, "\pi"] \arrow[d, "\iota_{\Pi'}"'] & |\Pi| \arrow[d, "\iota_{\Pi}"] \\
N_{\mathbb{R}}^{\Pi'} \arrow[r, hook] & N_{\mathbb{R}}^{\Pi}
\end{tikzcd}
$$

is commutative. Note that by linearity, it suffices to show that the restriction to the rays

$$
\begin{tikzcd}
\Pi'(1) \arrow[r, "\pi"] \arrow[d, "\iota_{\Pi'}"'] & \Pi(1) \arrow[d, "\iota_{\Pi}"] \\
N_{\mathbb{R}}^{\Pi'} \arrow[r, hook] & N_{\mathbb{R}}^{\Pi}
\end{tikzcd}
$$

commutes. In order to prove this, let $i\colon |\Pi| \to |\Pi|'$ be the natural inclusion and note that $\pi \circ i = \mathrm{Id}$ holds true. Now, for $\tau' \in \Pi'(1)$ we have two cases: either τ' is a ray in $\Pi(1)$ or it is not. If it is, then we have $\tau' = i(\tau)$ for some $\tau \in \Pi(1)$ and hence $\pi(\tau') = \tau$. Hence, for every $f \in M^{\Pi}\ \big(= (N^{\Pi})^{\vee}\big)$ we get

$$\iota_{\Pi'}(\tau')(f) = \mathrm{ord}_{B_{\tau'}}(f) = \mathrm{ord}_{B_{\tau}}(f) = \iota_{\Pi}(\tau)(f) = (\iota_{\Pi} \circ \pi)(\tau')(f).$$

For the case that τ' is not in $\Pi(1)$, note that $\pi(\tau') = 0$. Moreover, the support $|B_{\tau'}|$ is not contained in B and hence for all $f \in \Gamma\big(X \setminus B, \mathcal{O}_X^{\times}\big)$ we have that $\mathrm{ord}_{B_{\tau'}}(f) = 0$. Thus, we get that

$$\iota_{\Pi'}(\tau')(f) = \mathrm{ord}_{B_{\tau'}}(f) = 0 = \pi(\tau') = (\iota_{\Pi} \circ \pi)(\tau')(f).$$

This concludes the proof of the proposition. ■

Toroidal modifications

Recall from the classical theory of toric varieties that given a toric variety X_Σ corresponding to a fan Σ, there is a bijective correspondence between proper birational toric morphisms to X_Σ and subdivisions of the fan Σ. We will see that a similar phenomenon occurs in the toroidal case. We will describe a special class of proper birational morphisms between toroidal embeddings which correspond to projective subdivisions of the corresponding conical complex (Definition 3.1.8).

Definition 4.1.19. Let $U_{X_1} \hookrightarrow X_1$ and $U_{X_2} \hookrightarrow X_2$ be two toroidal embeddings and let $f\colon X_1 \to X_2$ be a birational morphism mapping U_{X_1} to U_{X_2}. Then f is called *toroidal* if for every closed point $x_1 \in X_1$ there exist local models (X_{σ_1}, x_1') at $x_1 \in X_1$ and (X_{σ_2}, x_2') at $f(x_1) \in X_2$, and a toric morphism $g\colon X_{\sigma_1} \to X_{\sigma_2}$, with $f(x_1') = x_2'$, such that the following diagram commutes:

$$
\begin{tikzcd}
\hat{\mathcal{O}}_{X_1,x_1} \arrow[r, "\simeq"] & \hat{\mathcal{O}}_{X_{\sigma_1},x_1'} \\
\hat{\mathcal{O}}_{X_2,x_2} \arrow[r, "\simeq"] \arrow[u, "\hat{f}^{\#}"] & \hat{\mathcal{O}}_{X_{\sigma_2},x_2'} \arrow[u, "\hat{g}^{\#}"']
\end{tikzcd}
$$

Here, $\hat{f}^{\#}$ and $\hat{g}^{\#}$ are the ring homomorphisms induced by f and g, respectively.

Remark 4.1.20. The following two properties are satisfied:

(1) The composition of two birational toroidal morphisms is again a birational toroidal morphism.

(2) A toroidal morphism $f\colon (U_{X_1} \hookrightarrow X_1) \to (U_{X_2} \hookrightarrow X_2)$ induces a morphism $\Pi_f\colon \Pi_{X_1} \to \Pi_{X_2}$ of conical complexes. The restrictions of Π_f to the cones of Π_{X_1} are dual to pulling back Cartier divisors. From this, we see that Π_f can also be considered as a morphism between weakly embedded conical complexes by adding to it the data of the linear map $N^{\Pi_{X_1}} \to N^{\Pi_{X_2}}$ dual to the pullback $\Gamma\left(U_{X_2}, \mathcal{O}_{X_2}^{\times}\right) \to \Gamma\left(U_{X_1}, \mathcal{O}_{X_1}^{\times}\right)$.

The following definition is taken from [KKMSD73, Definition 1, pg. 73].

Definition 4.1.21. A toroidal birational morphism $f\colon (U \hookrightarrow Y) \to (U \hookrightarrow X)$ between two toroidal embeddings is called *canonical over X* if the following conditions hold true:

(1) The diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] & & X \\ & U \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

is commutative.

(2) For all $x_1, x_2 \in X$ in the same stratum S and for all morphisms

$$\xi\colon \hat{\mathcal{O}}_{X,x_1} \longrightarrow \hat{\mathcal{O}}_{X,x_2} \tag{4.2}$$

which preserve the strata (i.e. if $S \subseteq \overline{S'}$ for some stratum S' then ξ takes the ideal of $\overline{S'}$ at x_1 to the ideal of $\overline{S'}$ at x_2), we have that $\mathrm{Spec}(\xi)$ can be lifted to give an isomorphism $Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_2}) \simeq Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_1})$ preserving the strata, i.e. such that the diagram

$$\begin{tikzcd} Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_2}) \arrow[r, "\simeq"] \arrow[d] & Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_1}) \arrow[d] \\ \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_2}) \arrow[r, "\mathrm{Spec}(\xi)"] & \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_1}) \end{tikzcd}$$

commutes.

We can now define the class of toroidal birational morphisms which correspond to decompositions of the conical complex. The following is [KKMSD73, Definition 3, pg. 87].

Definition 4.1.22. Consider a toroidal birational morphism $f\colon (U \hookrightarrow Y) \to (U \hookrightarrow X)$ forming a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] & & X \\ & U \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

and satisfying the following two conditions:

(1) Y has an open covering $\{V_i\}$ such that $U \subseteq V_i$, $f(V_i) \subseteq \mathrm{Star}(S_i)$ for some stratum S_i of X and V_i is affine and canonical over $\mathrm{Star}(S_i)$.

(2) Y is normal.

Toroidal embeddings $U \hookrightarrow Y$ as above are called *allowable modifications* of the toroidal embedding $U \hookrightarrow X$.

The following important theorem follows from [KKMSD73, Theorem 6*, 8*].

Theorem 4.1.23. *Given a toroidal embedding $U \hookrightarrow X$, there is a bijective correspondence between decompositions or subdivisions of the conical complex Π_X and isomorphism classes of allowable modifications or of proper allowable modifications of X, respectively.*

For the rest of this section we fix a smooth toroidal embedding $U \hookrightarrow X$ with associated smooth conical complex Π_X. We now see that one can further restrict to a more special kind of proper allowable modifications of X.

Definition 4.1.24. Let $\mathcal{F}$ be a coherent sheaf of fractional ideals on X. $\mathcal{F}$ is called *toroidal* if it is invariant under all isomorphisms $\xi\colon \hat{\mathcal{O}}_{X,x_1} \simeq \hat{\mathcal{O}}_{X,x_2}$ preserving the strata as in (4.2).

One can associate to any toroidal fractional ideal sheaf $\mathcal{F}$ a function $\phi_{\mathcal{F}}\colon |\Pi_X| \to \mathbb{R}$ satisfying the following properties (note the similarities with the properties given in Definition 3.1.8):

(1) $\phi_{\mathcal{F}}$ is conical.

(2) $\phi_{\mathcal{F}}$ is continuous and piecewise linear.

(3) $\phi_{\mathcal{F}}$ is rational, i.e. $\phi_{\mathcal{F}}|_{\sigma^S}$ belongs to $M^S_{\mathbb{Q}}$ for all strata S.

(4) $\phi_{\mathcal{F}}$ is concave on each cone σ^S for all strata S.

This is done as follows: for every stratum S, the restriction of $\mathcal{F}$ to the star of this stratum has the form

$$\mathcal{F}|_{\mathrm{Star}(S)} \simeq \bigotimes_{i=1}^{k} \mathcal{O}_X(-B_i)$$

for some $B_1, \dots, B_k \in M^S$. This allows us to make the following definition.

Definition 4.1.25. Let $\mathcal{F}$ be as above. The function $\phi_{\mathcal{F}}\colon |\Pi_X| = \left|\bigsqcup_S \sigma^S\right| \to \mathbb{R}$ is defined by the assignment

$$\sigma^S \ni v \longmapsto \min_{1\le i\le k} \langle B_i, v\rangle. \tag{4.3}$$

Remark 4.1.26. In [KKMSD73, pg. 91] it is shown that this definition is independent of the choice of the B_i's and that the function $\phi_{\mathcal{F}}$ indeed satisfies the properties listed above. Moreover, the function $\phi_{\mathcal{F}}$ has non-negative values if and only if $\mathcal{F}$ is an ideal sheaf.

Definition 4.1.27. As a converse to Definition 4.1.25, given a function $f\colon |\Pi_X| \to \mathbb{R}$ satisfying the properties (1)–(4) above, one can define a toroidal sheaf of fractional ideals $\mathcal{F}_f$ in the following way: for each stratum S, we let

$$\mathcal{F}_f|_{\mathrm{Star}(S)} \coloneqq \bigotimes_{\substack{B\in M^S \\ B\ge f \text{ on } \sigma^S}} \mathcal{O}_{\mathrm{Star}(S)}(-B).$$

These patch together giving a toroidal fractional ideal sheaf $\mathcal{F}_f$ in X.

The following theorem ([KKMSD73, Theorem 11*]) gives a way to construct special proper allowable modifications from functions on the conical complex as above. These are called *projective*.

Theorem 4.1.28. *Let $\mathcal{F}$ be a toroidal sheaf of fractional ideals and let $\phi_{\mathcal{F}}\colon |\Pi_X| \to \mathbb{R}$ be the corresponding function from Definition 4.1.25. Let $B_{\mathcal{F}}(X)$ be the normalized blow up of X along $\mathcal{F}$. Then the commutative diagram*

$$\begin{tikzcd} B_{\mathcal{F}}(X) \arrow[rr, "\pi"] & & X \\ & U \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

is a proper allowable modification of X described by subdividing the cones of $|\Pi_X|$ into the largest subcones on which $\phi_{\mathcal{F}}$ is linear. Proper allowable modifications of toroidal embeddings which arise in this way are called projective.

As a consequence, we get the following corollary.

Corollary 4.1.29. *There is a bijective correspondence between the set of isomorphism classes of projective allowable modifications of X and the set of projective subdivisions of the conical complex Π_X (Definition 3.1.8).*

4.2 Toroidal *b*-divisors

Throughout this section $U \hookrightarrow X$ will denote a smooth toroidal embedding with associated smooth, weakly embedded conical complex Π_X. The aim of this section is to define toroidal *b*-divisors on X and to give their combinatorial analogues. Specifically, we will see that one can view toroidal *b*-divisors on X as $\mathbb{Q}$-valued functions on the set $|\Pi_X|_{\mathbb{Q}}$ (Proposition 4.2.6).

We start by giving some definitions and results from the classical theory of toroidal embeddings. We refer to [KKMSD73] for further details and proofs. Note that many of these are direct generalizations of the toric case.

Definition 4.2.1. Let $\operatorname{Div}(X)_{\mathbb{Q}}$ be the group of $\mathbb{Q}$-Cartier divisors on X. We define the subgroup $\operatorname{Div}_0(X)_{\mathbb{Q}} \subseteq \operatorname{Div}(X)_{\mathbb{Q}}$ consisting of $\mathbb{Q}$-Cartier divisors which are supported on the boundary $X \setminus U$. Elements in $\operatorname{Div}_0(X)_{\mathbb{Q}}$ are called *toroidal divisors*.

Let $D \in \operatorname{Div}_0(X)_{\mathbb{Q}}$ be a toroidal divisor. By definition, D can also be seen as a toroidal sheaf of fractional ideals and hence, from Definition 4.1.25, we get a function $\phi_D\colon |\Pi_X| \to \mathbb{R}$ satisfying the properties (1)–(4) given there. However, in this case, for every stratum S, if we denote by X_{σ^S} the corresponding toric chart (see Lemma 4.1.8), then D restricted to $\operatorname{Star}(S)$ corresponds to a toric divisor in X_{σ^S}. Hence, the restriction of ϕ_D to a stratum is given by the *linear* virtual support function associated to a toric Cartier divisor on X_{σ^S} (see Section 1.1). Hence, property (4) can be replaced with the property of being linear on each cone σ^S. We summarize this in the following proposition, which is a special case of [KKMSD73, Theorem 9*].

Proposition 4.2.2. *Let $D \in \operatorname{Div}_0(X)_{\mathbb{Q}}$ be a toroidal divisor. Then the function $\phi_D\colon |\Pi_X| \to \mathbb{R}$ of Definition 4.1.25 satisfies the following properties:*

(1) ϕ_D is conical.

(2) ϕ_D is continuous.

(3) ϕ_D is linear on each cone σ^S for all strata S.

(4) ϕ_D is rational, i.e. $\phi_D|_{\sigma^S}$ belongs to $M^S_{\mathbb{Q}}$ for all strata S.

Hence, ϕ_D can be seen as a $\mathbb{Q}$-Cartier divisor on Π_X (Definition 3.2.4). Conversely, given a $\mathbb{Q}$-Cartier divisor ϕ on Π_X, the toroidal ideal sheaf D_ϕ of Definition 4.1.27 defines a toroidal divisor. Hence, the map

$$\mathrm{Div}_0(X)_{\mathbb{Q}} \longrightarrow \mathrm{Div}(\Pi_X)_{\mathbb{Q}} \tag{4.4}$$

given by the assignment

$$D \longmapsto \phi_D$$

induces a bijective correspondence between the set of toroidal divisors on X and the set of $\mathbb{Q}$-Cartier divisors on the conical complex Π_X, or equivalently, between toroidal divisors on X and conical, $\mathbb{Q}$-valued functions on the 1-dimensional skeleton of the conical complex Π_X.

Definition 4.2.3. A toroidal divisor D on X is said to be *combinatorially principal* (abbreviated cp) if its corresponding $\mathbb{Q}$-Cartier divisor ϕ_D on Π_X has this property (Definition 3.2.6). The subgroup of cp toroidal divisors is denoted by $\mathrm{cpDiv}_0(X)_{\mathbb{Q}}$.

We make the following remarks.

Remark 4.2.4. (1) In the non-toric case, a toroidal divisor on a combinatorial open subset of the form $\mathrm{Star}(S^\sigma)$ for some $\sigma \in \Pi_X$ is not necessarily principal. In case it is, it is defined by a rational function in $\Gamma\left(U, \mathcal{O}_X^\times\right)$. This means that the associated linear function on σ is the pullback of a function on M^{Π_X}. Hence, cp toroidal divisors correspond to toroidal divisors which are principal on open subsets of the above form.
(2) By Lemma 3.2.9, the weakly embedded conical complex Π_X is cp if and only if any toroidal divisor on X is cp.

We now come to the definition of toroidal b-divisors.

Definition 4.2.5. Consider the directed set $R(\Pi_X)$ of Definition 3.1.7. Moreover, for every subdivision $\Pi' \in R(\Pi_X)$, we denote by $X_{\Pi'}$ the corresponding proper allowable modification from Theorem 4.1.23. The *toroidal Riemann–Zariski space* of X is defined as the inverse limit

$$\mathfrak{X}_{\Pi_X} := \varprojlim_{\Pi' \in R(\Pi_X)} X_{\Pi'}$$

with maps given by the proper toroidal birational morphisms $X_{\Pi''} \to X_{\Pi'}$ induced whenever $\Pi'' \geq \Pi'$. The group of *toroidal Weil b-divisors* on X is defined to be the inverse limit

$$\mathrm{We}(\mathfrak{X}_{\Pi_X})_{\mathbb{Q}} := \varprojlim_{\Pi' \in R(\Pi_X)} \mathrm{Div}_0\left(X_{\Pi'}\right)_{\mathbb{Q}},$$

with maps given by the push-forward map of toroidal $\mathbb{Q}$-Cartier divisors. As in the toric case, we will write toroidal b-divisors in bold notation $\boldsymbol{D}$ to distinguish them from classical toroidal divisors D. We will refer to a toroidal Weil b-divisor just as as toroidal b-divisor.

We have the following combinatorial characterization of toroidal b-divisors.

Proposition 4.2.6. *There is a bijective correspondence between toroidal b-divisors and the set of conical, rational, $\mathbb{Q}$-valued functions on the set*

$$|\Pi_X|_{\mathbb{Q}} \coloneqq \bigsqcup_{S\,\text{stratum}} \sigma^S \otimes_{N^S} \mathbb{Q}.$$

Proof. This is a direct consequence of Proposition 4.2.2 and Theorem 4.1.23. ∎

4.3 Top intersection numbers of toroidal divisors

Throughout this section $U \hookrightarrow X$ will denote a smooth toroidal embedding with associated smooth, weakly embedded conical complex Π_X. In this section, we recall the definition of the tropicalization of an algebraic cycle class on X as is explained in [Gro15, Section 4.2] and we see that one can compute top intersection numbers of toroidal divisors using the tropical intersection product defined in Section 3.2.

For $0 \leq k \leq n$ we denote by $Z_k(X)_{\mathbb{Q}}$ the group of algebraic k-cycles on X with rational coefficients. For all $C \in Z_k(X)_{\mathbb{Q}}$, as is explained in [Gro15, Section 4.2], we can find a smooth subdivision $\Pi' \in R(\Pi_X)$ such that the map

$$\mathrm{trop}(C)\colon \Pi'(k) \longrightarrow \mathbb{Q},$$

given by the assignment

$$\sigma \longmapsto \deg\left([\pi^*(C)] \cdot \left[\overline{S^\sigma}\right]\right),$$

where $\pi\colon X_{\Pi'} \to X$ denotes the toroidal morphism induced by the subdivision $\Pi' \in R(\Pi_X)$, defines a k-dimensional Minkowski weight. Moreover, this Minkowski weight is independent of the choice of refinement Π' and thus we can make the following definition.

Definition 4.3.1. With notations as above, the map

$$\mathrm{trop}\colon Z_k(X)_{\mathbb{Q}} \longrightarrow Z_k(\Pi_X)$$

given by

$$C \longmapsto \mathrm{trop}(C)$$

is called the *tropicalization map*. In particular, $\mathrm{trop}(X)$ is the tropical cycle represented by the Minkowski weight in $M_n(\Pi_X)$ defined by taking weight one on all n-dimensional cones of Π_X. We denote this tropical cycle by $[\Pi_X]$.

We make the following remarks.

Remark 4.3.2. (1) There exists a subdivision $\Pi' \in R(\Pi_X)$ such that Π' is balanced (Definition 3.2.14). For the rest of this section we assume that Π_X is balanced.
(2) The tropicalization map factors through the group of numerical classes of k-cycles on X, which is denoted by $N_k(X)$, and hence we get a well defined tropicalization map

$$\mathrm{trop}\colon N_k(X)_{\mathbb{Q}} \longrightarrow Z_k(\Pi_X)$$

which we also denote by trop.

A natural question to ask is whether positivity notions on the algebraic and on the combinatorial side coincide. We make the following remark.

Remark 4.3.3. The tropicalization of an effective cycle is not necessarily a positive tropical cycle as in Definition 3.3.1. Indeed, consider for example the tropicalization of the exceptional divisor of the blow up of $\mathbb{P}^2$ at a torus fixed point.

An interesting question is the following: do effective cycle classes which lie in the interior of the pseudo effective cone $\overline{\mathrm{Eff}}_*(X) \subseteq N_*(X)_{\mathbb{R}}$ get mapped to positive cycles via the tropicalization map? These are the so called *big* cycle classes.

Moreover, if the answer to the above question is affirmative, then which numerical classes contained in the boundary of the pseudo effective cone are mapped to positive tropical cycles via the tropicalization map?

The following theorem relates algebraic and tropical intersections in the cp case and is essential for our results.

Theorem 4.3.4. *Let $D \in \mathrm{cpDiv}_0(X)_{\mathbb{Q}}$ be a cp toroidal divisor and let $[D]$ be its numerical equivalence class in $N_{n-1}(X)_{\mathbb{Q}}$. Then for every k-dimensional cycle class $[C]$ in $N_k(X)_{\mathbb{Q}}$ the following tropical cycle classes agree*

$$\mathrm{trop}\left([D]\cdot[C]\right) = [\phi_D]\cdot \mathrm{trop}(C),$$

where on the right hand side, the class $[\phi_D]$ is seen as an element in $\mathrm{cpCl}(\Pi_X)_{\mathbb{Q}}$ (Definitions 3.2.4 and 3.2.11).

Proof. This follows from [Gro15, Proposition 4.17]. ■

The following corollary is one of the main results of this section.

Corollary 4.3.5. *Let $D_1,\dots,D_n \in \mathrm{Div}_0(X)_{\mathbb{Q}}$ be a collection of (not necessarily cp) toroidal divisors on X. If the boundary divisor $B = X\setminus U$ is nc, then the algebraic top intersection number $D_1\cdots D_n$ can be computed tropically on a conical complex Π'_X as*

$$D_1\cdots D_n = \pi^*\phi_{D_1}\cdots\pi^*\phi_{D_n},$$

where Π'_X is the cp conical complex corresponding to the modified toroidal structure X' of X given in Proposition 4.1.18 and $\pi\colon \Pi'_X \to \Pi_X$ is the induced morphism of weakly embedded conical complexes.

Proof. By part (2) of Remark 4.2.4, the pullbacks $\pi^*\phi_{D_i}$ on Π'_X of the $\mathbb{Q}$-Cartier divisors ϕ_{D_i} corresponding to D_i, for $i = 1,\dots,n$, are cp. Hence, by Theorem 4.3.4 we get

$$D_1\cdots D_n|_X = D_1\cdots D_n|_{X'} = \pi^*\phi_{D_1}\cdots\pi^*\phi_{D_n}\cdot[\Pi'_X],$$

like we wanted to show. ■

We make to following remark.

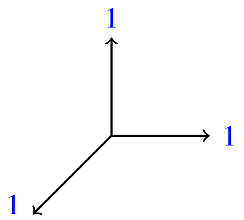
Remark 4.3.6. By the definition of the push-forward map of tropical cycles, we can see that $\pi_*[\Pi'_X] = [\Pi_X]$. Hence, in particular, if $D_1,\dots,D_n \in \mathrm{cpDiv}_0(X)_{\mathbb{Q}}$ are already cp toroidal divisors, then using the projection formula for the tropical intersection product (Proposition 3.2.21) and Theorem 4.3.4, we get

$$D_1\cdots D_n = \phi_{D_1}\cdots\phi_{D_n}\cdot[\Pi_X] = \phi_{D_1}\cdots\phi_{D_n}\cdot\pi_*[\Pi'_X] = \pi_*\left(\pi^*\phi_{D_1}\cdots\pi^*\phi_{D_n}\cdot[\Pi'_X]\right),$$

which combined with $\pi_*\left(Z_0(\Pi'_X)\right) = Z_0(\Pi_X) = \mathbb{Q}$ gives back Corollary 4.3.5.

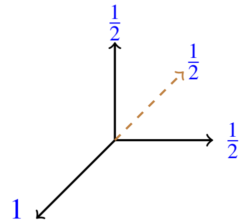
Example 4.3.7. Consider the following three 2-dimensional toric examples. These correspond to the first incarnations of the nef toric b-divisor $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ given by the concave function $\phi(a,b) = \frac{ab}{a+b}$ on $\mathbb{R}_{\geq 0}$ and $\phi(a,b) = \min(0,a,b)$ otherwise. The labels in blue on the rays correspond to the values of the Minkowski weight $D_{\Sigma_i} \cdot [\Sigma_i] \in M_1(\Sigma_i)$ for $i = 1,2,3$.

$X_{\Sigma_1} = \mathbb{P}^2$



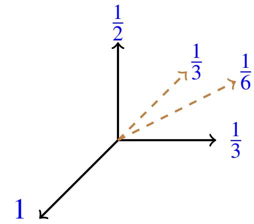
$D_{\Sigma_1} = -D_{(-1,-1)}$

$X_{\Sigma_2} = \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$



$D_{\Sigma_2} = -D_{(-1,-1)} + \frac{1}{2}D_{(1,1)}$

$X_{\Sigma_3} = \mathrm{Bl}_{\mathrm{pt}}\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$



$D_{\Sigma_3} = -D_{(-1,-1)} + \frac{1}{2}D_{(1,1)} - \frac{2}{3}D_{(2,1)}$

Hence, we get the top intersection number

$$D_{\Sigma_1}^2 = D_{\Sigma_1} \cdot (D_{\Sigma_1} \cdot [\Sigma_1]) = \sum_{\tau \in \Sigma_1(1)} \phi_{D_{\Sigma_1}}(\tau) \cdot (D_{\Sigma_1} \cdot [\Sigma_1])(\tau) = 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1 = 1.$$

Similarly, we obtain the other two top intersection numbers as

$$D_{\Sigma_2}^2 = 1 \cdot 1 + 0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4},$$
$$D_{\Sigma_3}^2 = 1 \cdot 1 + 0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{6} = \frac{13}{18}.$$

One can check that these numbers indeed agree with twice the volume of the polytopes $P_{D_{\Sigma_i}}$, which correspond to the algebraic top intersection numbers (see Section 1.1).

4.4 Integrability of nef toroidal b-divisors

Throughout this section $U \hookrightarrow X$ will denote a smooth toroidal embedding with associated smooth, weakly embedded conical complex Π_X. We will assume that Π_X is balanced and cp. The goal of this section is to prove an integrability result for nef toroidal b-divisors. This is done as follows: first we show that the set of $\mathbb{Q}$-Cartier divisors $\{\phi_D\}$ on Π_X, where D is a nef toroidal divisor on X, form a tropically nef collection $\mathcal{C}$ on Π_X (Definition 3.3.2). Then, generalizing the toric situation, we show that a nef toroidal b-divisor $\boldsymbol{D}$ induces a continuous function $\phi_{\boldsymbol{D}}$ on the whole of $|\Pi_X|$ such that the restriction to each cone is concave. Hence, $\phi_{\boldsymbol{D}}$ defines a continuous tropically nef b-divisor on Π_X (Definition 3.3.4) to which we can associate the limit measure $\mu_{\phi_{\boldsymbol{D}}}$ of Theorem 3.3.24. We define the mixed degree of a collection of n toroidal b-divisors $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ as a limit of top intersection numbers and we show that, in the nef case, this mixed degree exists, is finite and can be computed as an integral with respect to the mixed limit measure $\mu_{\phi_{\boldsymbol{D}_1}, \ldots, \phi_{\boldsymbol{D}_n}}$ associated to the continuous tropically nef b-divisors $\phi_{\boldsymbol{D}_i}$ on Π_X for $i = 1, \ldots, n$ (Definition 3.3.27). As a corollary, we give a formula for the difference of the self intersection number of any (not necessarily toroidal) b-divisor on a smooth toroidal embedding and its incarnation to X.

We start with the following positivity lemma.

Lemma 4.4.1. *The set of $\mathbb{Q}$-Cartier divisors $\{\phi_D\}$ on Π_X, where D is a nef toroidal divisor on X, form a tropically nef collection $\mathcal{C}$ on Π_X (Definition 3.3.2).*

Proof. Let $\mathcal{C} := \left\{\phi_D \,\middle|\, D \text{ nef toroidal divisor on } X \right\}$. We have to show that $\mathcal{C}$ satisfies the two properties given in Definition 3.3.2. Property (2) is a classical fact. To prove property (1), let $\ell_1, \dots, \ell_r$ be any tuple of non-negative integers such that $n - \sum_{i=1}^r \ell_i \geq 0$ and let $\phi_{D_1}, \dots, \phi_{D_r}$ be elements in $\mathcal{C}$. By Kleinmann's criterion for nefness, for all $i = 1, \dots, r$, the inequality $D_1^{\ell_1} \cdots D_r^{\ell_r} \cdot V \geq 0$ holds true for every subvariety V of codimension $\sum_{i=1}^r \ell_i$. Moreover, the restriction of D_i to V is still nef (see [Laz04a] for these two facts). Therefore, for every $\tau \in \Pi_X(n - \sum_{i=1}^r \ell_i)$, the weight of the tropical cycle $\phi_{D_1}^{\ell_1} \cdots \phi_{D_r}^{\ell_r} \cdot [\Pi_X]$ at τ satisfies

$$\begin{aligned}\left(\phi_{D_1}^{\ell_1} \cdots \phi_{D_r}^{\ell_r} \cdot [\Pi_X]\right)(\tau) &= \left(\phi_{D_1}^{\ell_1} \cdots \phi_{D_r}^{\ell_r} \cdot [\Pi_X]\right)|_{\Pi_X(\tau)} \\ &= \phi_{D_1}(\tau)^{\ell_1} \cdots \phi_{D_r}(\tau)^{\ell_r} \cdot [\Pi_X(\tau)] \\ &= \operatorname{trop}\left(\left(D_1|_{\overline{S^\tau}}\right)^{\ell_1} \cdots \left(D_r|_{\overline{S^\tau}}\right)^{\ell_r} \cdot \overline{S^\tau}\right) \geq 0,\end{aligned}$$

where the second and third equalities follow from Lemma 3.2.17 and Theorem 4.3.4, respectively. ■

Definition 4.4.2. A toroidal b-divisor $\boldsymbol{D} = (D_{\Pi'})_{\Pi' \in R(\Pi_X)}$ on X is said to be *nef* if for a cofinal subset $S \subseteq R(\Pi_X)$, the toroidal divisor $D_{\Pi'} \subseteq X_{\Pi'}$ is nef for all $\Pi' \in S$.

The following theorem describes nef toroidal b-divisors combinatorially.

Theorem 4.4.3. *Let $\boldsymbol{D}$ be a nef toroidal b-divisor on X. Then the corresponding function $\phi_{\boldsymbol{D}}$ on $|\Pi_X|_{\mathbb{Q}}$ of Proposition 4.2.6 extends to a continuous function on $|\Pi_X|_{\mathbb{R}}$, which we denote also by $\phi_{\boldsymbol{D}}$, such that the restriction of $\phi_{\boldsymbol{D}}$ to each cone $\sigma \in \Pi_X$ is concave. Hence, $\phi_{\boldsymbol{D}}$ defines a continuous tropically nef b-divisor on Π_X (Definition 3.3.4).*

Proof. Let $\boldsymbol{D}$ be a toroidal b-divisor and let $\sigma \in \Pi_X$ be a cone. Note that $\boldsymbol{D}|_{\mathrm{Star}(S^\sigma)}$ can be seen as a nef toric b-divisor on X_σ and thus, we have that the restriction $\phi_{\boldsymbol{D}}|_{\sigma \otimes \mathbb{Q}}$ is $\mathbb{Q}$-concave (see Lemma [CLS10, 9.2.1]). By the proof of Theorem 1.3.10, $\phi_{\boldsymbol{D}}|_{\sigma \otimes \mathbb{Q}}$ extends to a continuous, concave function $\phi_{\boldsymbol{D}}|_{\sigma \otimes \mathbb{R}}$. These extensions glue to a continuous function on the whole of $|\Pi_X|_{\mathbb{R}}$. Therefore, by Lemma 4.4.1, the function $\phi_{\boldsymbol{D}}$ arises as a limit of elements in a tropically nef collection $\mathcal{C}$ on Π_X and thus can be seen as a continuous tropically nef b-divisor. ■

We now come to the definition of the *mixed degree* of a collection of toroidal b-divisors.

Definition 4.4.4. Let $\boldsymbol{D}_1, \dots, \boldsymbol{D}_n$ be toroidal b-divisors on X. Their *mixed degree* is defined as the limit (in terms of nets) of top intersection numbers

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n := \lim_{\Pi' \in R(\Pi_X)} D_{1,\Pi'} \cdots D_{n,\Pi'},$$

provided the limit exists and is finite. If $\boldsymbol{D} = \boldsymbol{D}_1 = \cdots = \boldsymbol{D}_n$, then we call $\boldsymbol{D}^n$ the *degree* of the toroidal b-divisor $\boldsymbol{D}$. A toroidal b-divisor whose degree exists is called *integrable*.

The following is an integrability result for nef toroidal b-divisors. It is the main result of this chapter.

Theorem 4.4.5. *Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ be a collection of nef toroidal b-divisors on X. Then their mixed degree exists and is given by*

$$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n = \int_{\mathbb{S}^{\Pi_X}} \phi_{\boldsymbol{D}_1}(u) \mu_{\phi_{\boldsymbol{D}_2}, \ldots, \phi_{\boldsymbol{D}_n}},$$

where $\mu_{\phi_{\boldsymbol{D}_2}, \ldots, \phi_{\boldsymbol{D}_n}}$ is the mixed measure induced by a collection of continuous tropically nef b-divisors on Π_X (Definition 3.3.27).

In particular, if $\boldsymbol{D} = \boldsymbol{D}_1 = \cdots = \boldsymbol{D}_n$, then the degree of the nef b-divisor $\boldsymbol{D}$ exists and is given by

$$\boldsymbol{D}^n = \int_{\mathbb{S}^{\Pi_X}} \phi_{\boldsymbol{D}}(u) \mu_{\phi_{\boldsymbol{D}}}.$$

Proof. By Corollary 4.3.5 the mixed degree $\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$ can be computed as a limit of tropical intersection numbers. If they are moreover nef, then by Theorem 4.4.3 the corresponding functions $\phi_{\boldsymbol{D}_1}, \ldots, \phi_{\boldsymbol{D}_n}$ are continuous tropically nef b-divisors on Π_X. Hence, the theorem follows from Corollaries 3.3.25 and 3.3.28. ■

Remark 4.4.6. Just as in the toric case (Theorem 1.3.14) one can show that the difference $\boldsymbol{D}_1 - \boldsymbol{D}_2$ of nef toroidal b-divisors is integrable and that its degree can be computed as a sum of integrals with respect to mixed limit measures.

We can now compute the correction term of the self intersection number of a (not necessarily toroidal) b-divisor on a toroidal embedding assuming that its toroidal part is nef. Before stating the result, we give a definition.

Definition 4.4.7. A b-divisor on the toroidal embedding $U \hookrightarrow X$ is an element $\boldsymbol{D}$ in the inverse limit

$$\varprojlim_{\Pi' \in R(\Pi_X)} \operatorname{Div}(X_{\Pi'})_{\mathbb{Q}},$$

with maps given by the push-forward map of $\mathbb{Q}$-Cartier divisors on X. The degree of $\boldsymbol{D}$ is defined to be the limit (in the sense of nets)

$$\boldsymbol{D}^n := \lim_{\Pi' \in R(\Pi_X)} D^n_{\Pi'},$$

provided this limit exists and is finite.

Remark 4.4.8. Let $\boldsymbol{D} = (D_{\Pi'})_{\Pi' \in R(\Pi_X)}$ be a b-divisor on X. Suppose that on the level X we can decompose D_{Π_X} as

$$D_{\Pi_X} = C_{\Pi_X} + D'_{\Pi_X},$$

where C_{Π_X} intersects U and such that D'_{Π} is supported on the boundary $X \setminus U$. Then there exist a b-divisor $\boldsymbol{C} = (C_{\Pi'})_{\Pi' \in R(\Pi_X)}$ defined by $C_{\Pi'} = \pi^* C_{\Pi}$, where $\pi \colon X_{\Pi'} \to X$ denotes the corresponding allowable toroidal modification, and a toroidal b-divisor $\boldsymbol{D}' = \left(D'_{\Pi'}\right)_{\Pi' \in R(\Pi_X)}$ such that

$$D_{\Pi'} = C_{\Pi'} + D'_{\Pi'}.$$

holds for every $\Pi' \in R(\Pi_X)$.

We have the following Corollary.

Corollary 4.4.9. *Let $\boldsymbol{D}$ be a b-divisor on X and suppose that we have a decomposition*

$$\boldsymbol{D} = \boldsymbol{C} + \boldsymbol{D}'$$

as in Remark 4.4.8. Then, if $\boldsymbol{D}'$ is nef, the degree $\boldsymbol{D}^n$ of $\boldsymbol{D}$ exists, is finite, and is given by

$$\boldsymbol{D}^n = D^n_{\Pi_X} - D'^n_{\Pi_X} + \int_{\mathbb{S}^{\Pi_X}} \phi_{\boldsymbol{D}'}(u)\mu_{\phi_{\boldsymbol{D}'}}.$$

Proof. By assumption, we can write the incarnation D_{Π_X} of $\boldsymbol{D}$ on X as

$$D_{\Pi_X} = C_{\Pi_X} + D'_{\Pi_X},$$

where C_{Π_X} and D'_{Π_X} have the properties given in the statement of the corollary. Taking the n-th power on both sides, we get

$$D^n_{\Pi_X} = C^n_{\Pi_X} + D'^n_{\Pi_X} + \sum_{i=1}^{n-1} \binom{n}{i} C^{n-i}_{\Pi_X} D'^i_{\Pi_X}. \tag{4.5}$$

Now, let $\Pi' \in R(\Pi_X)$ and let $\pi \colon X_{\Pi'} \to X$ be the corresponding toroidal modification. Then for any number $i \in \{1, \dots, n-1\}$, by the projection formula, we have that

$$\pi_*\left(C^{n-i}_{\Pi'} D'^i_{\Pi'}\right) = \pi_*\left(\pi^* C^{n-i}_{\Pi_X} D'^i_{\Pi'}\right) = C^{n-i}_{\Pi_X}\left(\pi_* D'^i_{\Pi'}\right) = C^{n-i}_{\Pi_X} D'^i_{\Pi_X}. \tag{4.6}$$

Hence, taking limits in (4.5) and observing (4.6), we use Theorem 4.4.5 and get

$$\begin{aligned}
\boldsymbol{D}^n &= \lim_{\Pi' \in R(\Pi_X)} D^n_{\Pi'} \\
&= \lim_{\Pi' \in R(\Pi_X)} \left(C_{\Pi'} + D'_{\Pi'}\right)^n \\
&= \lim_{\Pi' \in R(\Pi_X)} \left(C^n_{\Pi'} + D'^n_{\Pi'} + \sum_{i=1}^{n-1} \binom{n}{i} C^{n-i}_{\Pi'} D'^i_{\Pi'}\right) \\
&= C^n_{\Pi_X} + \boldsymbol{D}'^n + \sum_{i=1}^{n-1} \binom{n}{i} C^{n-i}_{\Pi_X} D'^i_{\Pi_X} \\
&= \left(C_{\Pi_X} + D'_{\Pi_X}\right)^n - D'^n_{\Pi_X} + \boldsymbol{D}'^n \\
&= D^n_{\Pi_X} - D'^n_{\Pi_X} + \boldsymbol{D}'^n \\
&= D^n_{\Pi_X} - D'^n_{\Pi_X} + \int_{\mathbb{S}^{\Pi_X}} \phi_{\boldsymbol{D}'}(u)\mu_{\phi_{\boldsymbol{D}'}},
\end{aligned}$$

concluding the proof of the corollary. ∎

Remark 4.4.10. It follows from Section 3.4 that in the case of a fan or in the case of dimension 2, we may use the mixed surface area measure associated to convex bodies to compute degrees of nef toroidal b-divisors. Hence, in either of these cases, if the restrictions of the toroidal b-divisors to the cones are of class C^2, then we can compute mixed degrees of toroidal b-divisors explicitly as Lebesgue integrals of determinants of Hessians using Corollary 3.4.18 and Proposition 3.4.22.

Chapter 5

The b-line bundle of Jacobi forms on the universal elliptic curve

As an application of the intersection theory of b-divisors on toroidal embeddings developed so far, we compute the degree of the b-divisor of Jacobi forms of weight k and index m with respect to the principal congruence subgroup $\Gamma(N) \subseteq \mathrm{PSL}_2(\mathbb{Z})$ on the generalized universal elliptic curve. Moreover, we give the definition of a b-line bundle so that we can talk about the b-line bundle of Jacobi forms of weight k and index m with respect to $\Gamma(N)$. This definition is taken from a (still unpublished) article by M. Jespers and R. de Jong. We also prove three statements which show that it is meaningful to consider the b-divisorial approach instead of just fixing one (canonical) compactification. First, we give a geometrical interpretation for the space of global sections of the b-line bundle of Jacobi forms, namely we prove that there is an isomorphism between the space of global sections of the b-line bundle of Jacobi forms and the so called Jacobi cusp forms. Secondly, we show that Chern–Weil theory holds in this context. Finally, we give a Hilbert–Samuel formula relating the asymptotic growth of the dimension of the space of global sections of multiples of the b-line bundle of Jacobi forms with its degree. The latter two results were shown in [BKK16, Theorem 5.1, Theorem 5.2] for the case $k = m = 4$. This chapter can be seen as a generalization and a conceptual interpretation of the results given in this article.

5.1 The universal elliptic curve and the line bundle of Jacobi forms with its invariant metric

Throughout this chapter everything will be defined over the field of complex numbers $\mathbb{C}$. The variable N will denote an integer bigger or equal to 3 and $\Gamma(N) \subseteq \mathrm{PSL}_2(\mathbb{Z})$ will denote the principal congruence subgroup of level N.

In this section we recall the definition of the universal elliptic curve $E^0(N)$ of level N lying over the modular curve $X^0(N)$ of level N and its smooth toroidal compactifications $E(N)$ and $X(N)$, respectively. We also recall the definition of the line bundle $L_{k,m,N}$ of Jacobi forms of weight k and index m with respect to $\Gamma(N)$ and its translation invariant metric $||\cdot||_{\mathrm{Pet}}$. Most of the definitions and results of this section are taken from [Kra91] and [BKK16].

The universal elliptic curve

Let $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \tau = \xi + i\eta\,,\, \eta > 0\}$ be the upper half plane. It is known that the group $\Gamma(N)$ acts on $\mathbb{H}$ by fractional linear transformations via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) := \frac{a\tau + b}{c\tau + d},$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$, $\tau \in \mathbb{H}$, and that this action extends naturally to the extended upper half plane $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. The quotient space $X(N) := \Gamma(N) \setminus \mathbb{H}^*$ is called the *modular curve of level* N. It is the compactification of $X^0(N) := \Gamma(N) \setminus \mathbb{H}$ by adding $p_N := [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]\,/N$ so called *cusps*.

Remark 5.1.1. In the article [BKK16], the authors consider the index of $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ instead of in $\mathrm{PSL}_2(\mathbb{Z})$. Hence, their index differs from ours by a factor 2.

The semi-direct product $\Gamma(N) \ltimes \mathbb{Z}^2$ acts on $\mathbb{H} \times \mathbb{C}$ by the assignment

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right](\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right), \tag{5.1}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$ and $(\lambda, \mu) \in \mathbb{Z}^2$. Since $N \geq 3$, the group $\Gamma(N)$ is torsion-free and hence the action given in (5.1) is free. Therefore, the quotient space $E^0(N) := \Gamma(N) \ltimes \mathbb{Z}^2 \setminus \mathbb{H} \times \mathbb{C}$ defines a smooth complex surface together with a smooth surjective morphism

$$p^0 \colon E^0(N) \longrightarrow X^0(N),$$

with fiber the elliptic curve $(p^0)^{-1}([\tau]) = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$.

This surface is known to extend to a compact complex surface by using a toroidal compactification, and mapping surjectively onto $X(N)$. This compactified surface is called the *generalized universal elliptic curve*. It is denoted by $E(N)$ and the corresponding surjection by

$$p \colon E(N) \longrightarrow X(N).$$

The surface $E(N)$ can be explicitly described as follows: the fibers $p^{-1}(P_j)$ above the cusps $P_j \in X(N)$ for $j = 1, \ldots, p_N$ are N-gons, i.e. we have

$$p^{-1}(P_j) = \bigcup_{v=0}^{N-1} \Theta_{j,v},$$

where $\Theta_{j,v} \simeq \mathbb{P}^1$ is embedded into $E(N)$ with self intersection number -2, and otherwise

$$\Theta_{j,v} \cdot \Theta_{j',v'} = \begin{cases} 1, & \text{if } j = j' \text{ and } v - v' \equiv \pm 1 \bmod N, \\ 0, & \text{otherwise,} \end{cases}$$

where we consider the index v modulo N.

In local coordinates the situation above the cusp $P_1 = [\infty]$ can be described as follows: the irreducible component $\Theta_v := \Theta_{1,v} \subseteq E(N)$ can be covered by two affine charts $W_v^0, W_v^1 \subseteq E(N)$, where W_v^0 contains the point $\Theta_v \cap \Theta_{v+1}$ and W_v^1 contains the point $\Theta_v \cap \Theta_{v-1}$. Since Θ_v and Θ_{v+1}

intersect transversally, we may choose coordinates (u_ν, v_ν) on the chart W_ν^0 in such a way that $\Theta_\nu|_{W_\nu^0}$ is given by the equation $v_\nu = 0$ and $\Theta_{\nu+1}|_{W_\nu^0}$ by the equation $u_\nu = 0$. Using that the self intersection number $\Theta_\nu \cdot \Theta_\nu$ is -2 we obtain that the coordinates of W_ν^1 are given by $u_\nu^{-1}, u_\nu^2 v_\nu$. Moreover, since the open subset $W_{\nu+1}^1$ agrees with W_ν^0, we deduce the relations

$$u_{\nu+1} = v_\nu^{-1} \quad \text{and} \quad v_{\nu+1} = u_\nu v_\nu^2.$$

Furthermore, if we let $q := e^{2\pi i\tau/N}$ and $\zeta := e^{2\pi i z}$, then we have that

$$u_\nu v_\nu = q \quad \text{and} \quad u_\nu^{\nu+1} v_\nu^\nu = \zeta.$$

The embedding $E^0(N) \hookrightarrow E(N)$ is toroidal, hence from the previous chapter we know that it comes with a (non-connected) rational conical polyhedral complex which we denote by $\Pi_{E(N)}$. It consists of p_N connected components (one over each cusp), each of which consists of a union of N smooth 2-dimensional cones glued along their faces.

Note however that by our choice of $\Gamma(N)$, the quotient group $\mathrm{PSL}_2(\mathbb{Z})/\Gamma(N)$ acts transitively on the set of cusps, hence it will be sufficient to work only over the cusp $P_1 = [\infty]$.

Jacobi forms and the translation invariant metric

Let $k \in 2\mathbb{N}$ and $m \in \mathbb{N}$. We recall the definition of Jacobi forms of weight k and index m for $\Gamma(N)$ and the so called translation invariant metric on the line bundle of Jacobi forms.

Definition 5.1.2. A holomorphic function $f\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is called a *Jacobi form of weight k and index m for* $\Gamma(N)$, if it satisfies the following two properties:

(1) f satisfies the functional equation

$$\begin{aligned} &f\left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-k}\exp\left(2\pi i m\left[\lambda^2\tau+2\lambda z-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right]\right) \\ &= f(\tau,z), \end{aligned} \tag{5.2}$$

for all $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu)\right] \in \Gamma(N) \ltimes \mathbb{Z}^2$.

(2) At the cusp $P_1 = [\infty]$, the function f has a Fourier expansion of the form

$$f(\tau,z) = \sum_{\substack{n\in\mathbb{N}, r\in\mathbb{Z} \\ 4mn-Nr^2\geq 0}} c(n,r) q^n \zeta^r \tag{5.3}$$

and similar Fourier expansions at other cusps.

Notation. We use the same notation as in [BKK16]: the vector space of Jacobi forms of weight k and index m for $\Gamma(N)$ is denoted by $J_{k,m}(\Gamma(N))$. If condition (5.3) on the Fourier expansion is restricted to the summation over $n \in \mathbb{N}_{>0}$, and $r \in \mathbb{Z}$ such that $4mn - Nr^2 > 0$, then the function f is called a *Jacobi cusp form of weight k and index m for* $\Gamma(N)$. The vector space of Jacobi cusp forms of weight k and index m for $\Gamma(N)$ is denoted by $J_{k,m}^{\mathrm{cusp}}(\Gamma(N))$.
If condition (5.3) on the Fourier expansions is dropped, the function f is called a *weak Jacobi form of weight k and index m for* $\Gamma(N)$. The vector space of weak Jacobi forms of weight k and index m for $\Gamma(N)$ is denoted by $J_{k,m}^{\mathrm{weak}}(\Gamma(N))$.

Remark 5.1.3. One can check that the assignment

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu)\right] \longmapsto (c\tau+d)^{-k}\exp\left(2\pi i m\left[\lambda^2\tau+2\lambda z-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right]\right)$$

defines a 1-cocycle, and hence a 1-cohomology class of

$$H^1\left(\Gamma(N)\ltimes\mathbb{Z}^2,\mathbb{C}^\times\right)\simeq H^1\left(E^0(N),\mathcal{O}^\times_{E^0(N)}\right)\simeq \operatorname{Pic}\left(E^0(N)\right).$$

We denote by $L_{k,m,N}$ the corresponding line bundle on $E^0(N)$. By definition, the space of global sections of $L_{k,m,N}$ equals the space of weak Jacobi forms of weight k and index m for $\Gamma(N)$.

We will give an explicit equation defining a weak Jacobi form of weight k and index m for $\Gamma(N)$, which we denote by $G(\tau,z)$. Before we do this, we need the following two lemmas which can be found in [Kra91, Lemma 2.2 and Lemma 2.3].

Lemma 5.1.4. *The theta function $\theta_{1,1}:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ given by*

$$\theta_{1,1}(\tau,z) := \sum_{n\in\mathbb{Z}}\exp\left(\pi i\tau\left[n+\frac{1}{2}\right]^2+2\pi i\left[z+\frac{1}{2}\right]\left[n+\frac{1}{2}\right]\right),$$

satisfies the functional equation

$$\begin{aligned}&\theta_{1,1}\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-1/2}\exp\left(\pi i\left[\lambda^2\tau+2\lambda z-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}\right]\right)\\ &=\chi_1\theta_{1,1}(\tau,z)\end{aligned} \tag{5.4}$$

for all $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu)\right]\in \mathrm{PSL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2$*, with χ_1 an 8-th root of unity.*

Lemma 5.1.5. *The eta function $\eta:\mathbb{H}\to\mathbb{C}$ given by*

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right),$$

satisfies the functional equation

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-1/2}=\chi_2\eta(\tau) \tag{5.5}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{PSL}_2(\mathbb{Z})$*, with χ_2 a 24-th root of unity.*

Now, let $\chi := (\chi_1\chi_2^{-1})^{2m}$ and let g denote a modular form of weight k and character $\bar{\chi}$ with respect to $\Gamma(N)$. We recall that the degree of $\operatorname{div}(g)$ is given by the quantity $k\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]/12$ (see also [Kra91, pg. 5]). We can now define the weak Jacobi form $G(\tau,z)$ of weight k and index m for $\Gamma(N)$.

Proposition/Definition 5.1.6. *The function*

$$G(\tau,z) := \left(\frac{\theta_{1,1}(\tau,z)}{\eta(\tau)}\right)^{2m}g(\tau)$$

satisfies the functional equation (5.2) and hence defines a global section of the line bundle $L_{k,m,N}$.

Proof. This follows from the functional equations (5.4) and (5.5). ■

Remark 5.1.7. We make the following remarks.

(1) The spaces $J_{k,m}(\Gamma(N))$ and $J_{k,m}^{\mathrm{cusp}}(\Gamma(N))$ are finite-dimensional. Their dimensions are computed in [Kra91, Theorems 3.3, 3.4 and 3.8]. From these computations, we get the asymptotic formula

$$\dim J_{\ell k,\ell m}(\Gamma(N)) = \frac{\ell^2 mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{6} + o(\ell^2). \tag{5.6}$$

(2) The dimension of the vector space of Jacobi cusp forms is computed by giving an isomorphism

$$J_{k,m}^{\mathrm{cusp}}(\Gamma(N)) \simeq H^0(E(N),\mathcal{F}_{k,m,N}),$$

where $\mathcal{F}_{k,m,N}$ is a subsheaf of $j_*(L_{k,m,N})$. Here, $j\colon E^0(N) \hookrightarrow E(N)$ denotes the natural inclusion (see [Kra91, Theorem 2.6]). The construction of the subsheaf $\mathcal{F}_{k,m,N}$ is made ad-hoc and gives no geometric interpretation of the cusp condition of a Jacobi form. We will see that once we take into account the translation invariant metric, a geometric interpretation can be given. Indeed, we will see that we can define a b-line bundle of Jacobi forms of weight k and index m on $E(N)$ which takes into account the singularities of the invariant metric. This will give rise to an isomorphism between the space of Jacobi cusp forms and the space of global sections of this b-line bundle (Theorem 5.3.3).

Definition 5.1.8. The *Petersson translation invariant metric* $||\cdot||_{\mathrm{Pet}}$ on the line bundle $L_{k,m,N}$ is the natural smooth hermitian metric defined by

$$||f(\tau,z)||^2_{\mathrm{Pet}} = |f(\tau,z)|^2 \exp\left(-4\pi m \frac{\mathrm{Im}(z)^2}{\mathrm{Im}(\tau)}\right)\mathrm{Im}(\tau)^k, \tag{5.7}$$

for $(\tau,z)\in\mathbb{H}\times\mathbb{C}$. We denote by

$$\overline{L}_{k,m,N} := (L_{k,m,N}, ||\cdot||_{\mathrm{Pet}})$$

the corresponding hermitian line bundle, i.e. the line bundle together with the metric.

Remark 5.1.9. One checks that the metric given in (5.7) is indeed invariant under the action of $\Gamma(N)\ltimes\mathbb{Z}^2$. Hence, the above hermitian line bundle is well defined.

We end this section with the following lemma, which we cite from [BKK16, Lemma 2.10], describing the local behavior of the Petersson translation invariant metric over the cusp P_1.

Lemma 5.1.10. *Let notations be as above. Locally, in the affine chart W_v^0 over the cusp P_1, the hermitian metric $||\cdot||_{\mathrm{Pet}}$ is described by the formula*

$$\begin{aligned}\log(||f(\tau,z)||^2_{\mathrm{Pet}})|_{W_v^0} = &\log(|f(\tau,z)|^2)|_{W_v^0}\\ &+\frac{m}{N}\left((v+1)^2\log(u_v\bar{u}_v)+v^2\log(v_v\bar{v}_v)-\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v)}\right)\\ &+k\log\left(-\frac{N}{4\pi}(\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v))\right).\end{aligned}$$

5.2 The b-line bundle of Jacobi forms

Throughout this section, N will denote an integer bigger than or equal to 3 and $\Gamma(N)$ will denote the principal congruence subgroup of level N. As before, we write $X(N)$ for the compactified modular curve of level N and $E(N)$ for the generalized universal elliptic curve of level N. Its associated conical complex will be denoted by $\Pi_{E(N)}$. In this section we start by giving the definition of a b-line bundle and its relationship with b-divisors. We then recall the so called Mumford–Lear extension of a line bundle and its relationship with b-line bundles. We then study the Mumford–Lear extensions of the line bundle $L_{k,m,N}$ of Jacobi forms of weight k and index m for $\Gamma(N)$. We will see that the line bundle of Jacobi forms admits all Mumford–Lear extensions and hence we can define the b-line bundle of Jacobi forms and consider its corresponding b-divisor $\boldsymbol{D}_{k,m,N}$ which takes into account the singularities of the invariant metric along the boundary of all smooth toroidal compactifications of $E^0(N)$. Finally, using the intersection theory of toroidal b-divisors on toroidal varieties developed in Chapter 4, we compute the degree of $\boldsymbol{D}_{k,m,N}$. We take most of the definitions and results of this section from [BKK16, Sections 3 and 4]. The definition of a b-line bundle is taken from a (still unpublished) article by M. Jespers and R. de Jong.

b-line bundles

Let $U \hookrightarrow X$ be a smooth toroidal embedding of dimension n with corresponding weakly embedded conical complex Π_X. Recall the directed set $R(\Pi_X)$ given in Definition 3.1.7. We give the definition of the Picard group of the toroidal Riemann–Zariski space $\operatorname{Pic}(\mathfrak{X}_{\Pi_X})$. Before we give the definition, recall the definition of the Picard group of the toric Riemann–Zariski space $\operatorname{Pic}(\mathfrak{X}_\Sigma)$ in the case that $X = X_\Sigma$ is a toric variety (Definition 2.2.10). Here, given any proper birational map $X_{\Sigma''} \to X_{\Sigma'}$ in the projective system, we have a naturally defined push-forward map of line bundles by the fact that in this case, numerical equivalence of divisors agrees with linear equivalence. In general, this is no longer true. However, we can do the following: for any $\Pi' \in R(\Pi_X)$, we denote by $B' = X_{\Pi'} \setminus U$ the boundary divisor and we let B'_{sing} denote its singular locus. This is a set of codimension 2. Moreover, we set $U' = X_{\Pi'} \setminus B'_{\text{sing}}$.

Now, given $\Pi'' \geq \Pi' \in R(\Pi_X)$, we denote by $\pi\colon X_{\Pi''} \to X_{\Pi'}$ the corresponding proper toroidal birational morphism. Note that we have a canonical isomorphism $\pi|_{\pi^{-1}(U')}\colon \pi^{-1}(U') \to U'$ whose inverse map we denote by φ. Since $B''_{\text{sing}} \subseteq \pi^{-1}\left(B'_{\text{sing}}\right)$, we have that $\pi^{-1}(U') \subseteq U''$. We thus obtain a canonical immersion $\iota\colon U' \hookrightarrow U''$. Hence, since the sets B'_{sing} and B''_{sing} are of codimension 2, we obtain a natural push-forward map of line bundles

$$\pi_*\colon \operatorname{Pic}(X_{\Pi''}) \simeq \operatorname{Pic}(U'') \longrightarrow \operatorname{Pic}(U') \simeq \operatorname{Pic}(X_{\Pi'}) \tag{5.8}$$

induced by taking the pull-back of line bundles along ι.

Remark 5.2.1. The push-forward map described above is compatible with the push-forward map of Cartier divisors. Indeed, let D be a Cartier divisor on $X_{\Pi''}$. We set $\mathcal{L}'' = \mathcal{O}_{X_{\Pi''}}(D)$ and $\mathcal{L}' = \mathcal{O}_{X_{\Pi'}}(\pi_* D)$. Then we have that

$$\iota^*\mathcal{L}''|_{U''} = \iota^*\mathcal{L}''|_{\pi^{-1}(U')} = \mathcal{O}_{U'}(\varphi^*(D|_{\pi^{-1}(U')})) = \mathcal{O}_{U'}(\pi_*(D|_{\pi^{-1}(U')})) = \mathcal{L}'|_{U'}.$$

Definition 5.2.2. The Picard group of the toroidal Riemann–Zariski space $\operatorname{Pic}(\mathfrak{X}_{\Pi_X})$ is defined as the inverse limit

$$\operatorname{Pic}(\mathfrak{X}_{\Pi_X}) := \varprojlim_{\Pi' \in R(\Pi_X)} \operatorname{Pic}(X_{\Pi'}),$$

with maps given by the push-forward map of line bundles given in (5.8). Elements in $\mathrm{Pic}\,(\mathfrak{X}_{\Pi_X})$ are called *b-line bundles* on X.

We will denote b-line bundles in bold $\boldsymbol{\mathcal{L}}$ to distinguish them from classical line bundles $\mathcal{L}$.

Remark 5.2.3. It turns out that one can naturally identify the Picard group $\mathrm{Pic}\,(\mathfrak{X}_{\Pi_X})$ with the group of b-divisors on the toroidal embedding $U \hookrightarrow X$ (Definition 4.4.7) modulo the group of principal b-divisors, i.e. of the form b-div(f) for some non-zero rational function f on X. We write $\boldsymbol{\mathcal{L}} = \mathcal{O}_{E(N)}(\boldsymbol{D})$ for the b-line bundle corresponding to the b-divisor $\boldsymbol{D}$.

A section s of a b-line bundle $\boldsymbol{\mathcal{L}} = (\mathcal{L}_{\Pi'})_{\Pi' \in R(\Pi_X)}$ is defined to be a section of its restriction to U. If s is non-zero then on each $X_{\Pi'}$ we may consider $\mathrm{div}_{X_{\Pi'}}(s)$, the $\mathbb{Q}$-Cartier divisor of s seen as a rational section of $\mathcal{L}_{\Pi'}$. This induces a b-divisor which is denoted by b-div(s). We have $\boldsymbol{\mathcal{L}} = \mathcal{O}_{E(N)}(b\text{-div}(s))$.

Mumford–Lear extensions of a line bundle

Let X denote a smooth projective variety of dimension n together with a snc divisor $B \subseteq X$. Let $U = X \setminus B$. Then $U \hookrightarrow X$ defines a toroidal embedding whose associated weakly embedded conical complex Π_X is the cone over the Clemens complex associated to the pair (X, B) (see Example 4.1.13).

At this point the reader is referred to [BKK16, Section 3] for the various notions of growth for metrics and differential forms.

Definition 5.2.4. With notations as above, let $\overline{L} = (L, ||\cdot||)$ be a hermitian line bundle on U. We say that $\overline{L}$ *admits a Mumford–Lear extension to* X, if there is an integer $e \geq 1$, a line bundle $\mathcal{L}$ on X, an algebraic subset $T \subseteq X$ of codimension at least 2 that is contained in B, a smooth hermitian metric $||\cdot||$ on $\mathcal{L}|_U$ that has logarithmic growth along $B \setminus T$ and an isometry

$$\alpha\colon (L, ||\cdot||)^{\otimes e} \longrightarrow (\mathcal{L}|_U, ||\cdot||).$$

The 5-tuple $(e, \mathcal{L}, T, ||\cdot||, \alpha)$ is called a *Mumford–Lear extension* of $\overline{L}$.

If it is the case that $\overline{L}$ admits a Mumford–Lear extension, then we can attach the $\mathbb{Q}$-Cartier divisor $\mathrm{div}_X(s)$ on X to any rational section s of L by defining

$$\mathrm{div}_X(s) := \frac{1}{e}\mathrm{div}(\alpha(s^{\otimes e})) \in \mathbb{Q}\text{-Ca}(X).$$

This is well defined (see [BKK16, Corollary 3.10]). We now consider Mumford–Lear extensions on different birational models of X. We make the following definition.

Definition 5.2.5. The subset $R(\Pi_X)_{\mathrm{snc}} \subseteq R(\Pi_X)$ is defined by

$$R(\Pi_X)_{\mathrm{snc}} := \left\{\Pi' \in R(\Pi_X) \;\middle|\; X_{\Pi'} \setminus U \text{ is snc}\right\},$$

where $X_{\Pi'}$ denotes the allowable toroidal modification of X corresponding to Π' (see Theorem 4.1.23).

We can now define what it means for a line bundle to admit all Mumford–Lear extensions.

Definition 5.2.6. Let notations be as above. We say that the hermitian line bundle $\overline{L}$ *admits all Mumford–Lear extensions over X* if, for every object $\Pi' \in R(\Pi_X)_{\mathrm{snc}}$, the hermitian line bundle $\pi_{\Pi'}^*\overline{L}$ on $U_{\Pi'} := \pi_{\Pi'}^{-1}(U)$ admits a Mumford–Lear extension to $X_{\Pi'}$. Here, $\pi_{\Pi'} \colon X_{\Pi'} \to X$ denotes the induced allowable toroidal modification.

Any hermitian line bundle admitting all Mumford–Lear extensions defines a b-line bundle and thus a b-divisor by considering any rational section. This is given explicitly in the following way.

Definition 5.2.7. Assume that $\overline{L}$ admits all Mumford–Lear extensions over X and consider a non-zero rational section s of L. For every $\Pi' \in R(\Pi_X)_{\mathrm{snc}}$, let $(e', \mathcal{L}', T', ||\cdot||', \alpha')$ be a Mumford–Lear extension of $\pi_{\Pi'}^*\overline{L}$ to $X_{\Pi'}$. Then the tuple $(\mathcal{L}')_{\Pi' \in R(\Pi_X)_{\mathrm{snc}}}$ defines a b-line bundle $\boldsymbol{\mathcal{L}}$ on X and at each level Π' the corresponding divisor $\mathrm{div}_{X_{\Pi'}}(s)$ of s on $X_{\Pi'}$ is defined by

$$\mathrm{div}_{X_{\Pi'}}(s) = \frac{1}{e'}\mathrm{div}(\alpha'(s^{\otimes e'})) \in \mathbb{Q}\text{-}\mathrm{Ca}(X_{\Pi'}).$$

Thus, the b-divisor associated to s is given by

$$b\text{-}\mathrm{div}(s) := (\mathrm{div}_{X_{\Pi'}}(s))_{\Pi' \in R(\Pi_X)_{\mathrm{snc}}} \in \varprojlim_{\Pi' \in R(\Pi_X)_{\mathrm{snc}}} \mathbb{Q}\text{-}\mathrm{Ca}(X_{\Pi'}).$$

This is well defined (see [BKK16, Proposition 3.15]).

Mumford–Lear extensions of the line bundle of Jacobi forms

We now study the Mumford–Lear extensions of the line bundle of Jacobi forms. We start by recalling the following proposition from [BKK16, Proposition 4.1].

Proposition 5.2.8. *Let (n', n'') be a pair of coprime positive integers. Let (u, v) be coordinates of $\mathbb{C}^2$ and denote by $U \subseteq \mathbb{C}^2$ the open subset defined by $|u| < 1$, $|v| < 1$. Let $B \subseteq U$ be the normal crossing divisor of equation $uv = 0$. Let $f_{n',n''}$ be the function on U given by*

$$f_{n',n''}(u,v) = \frac{1}{n'n''}\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{n'\log(u_v\bar{u}_v) + n''\log(v_v\bar{v}_v)}.$$

This function satisfies the following properties:

(1) $f_{n',n''}$ is a pre-log-log function along $B \setminus \{(0,0)\}$.

(2) The equality

$$\bar{\partial} f_{n',n''} \wedge \partial\bar{\partial} f_{n',n''} = 0$$

holds true. The differential forms $f_{n',n''}, \partial f_{n',n''}, \bar{\partial} f_{n',n''}$ and $\partial\bar{\partial} f_{n',n''}$ and all products between them are locally integrable. Moreover, any product of $\partial\bar{\partial} f_{n',n''}$ with a pre-log-log form along B is also locally integrable.

(3) Let $\pi \colon U \to U$ be the map given by $(s,t) \mapsto (st, t)$. Note that this gives a chart of the blow up of U along $(0,0)$. Then

$$\pi^* f_{n',n''}(s,t) = \frac{1}{n'n''(n'+n'')}\log(t\bar{t}) + f_{n',n'+n''}(s,t).$$

Now, let us consider the toroidal embedding $E^0(N) \hookrightarrow E(N)$. Note that the boundary divisor $B = E(N) \setminus E^0(N)$ is snc. As we said before, the conical complex $\Pi_{E(N)}$ attached to the embedding consists of p_N connected components, each of which is a union of N smooth 2-dimensional cones glued along their faces. Each of the 2-dimensional cones corresponds to a double point of B. Let $T \subseteq B$ be the set of double points of B. This is a 2-codimensional set in $E(N)$. We also denote by $B_{\mathrm{smooth}} = B \setminus T$ the smooth locus of B and we let H be the divisor on $E(N)$ defined as the image of the zero section $X(N) \to E(N)$.

We will study Mumford–Lear extensions of the hermitian line bundle $\overline{L}_{k,m,N}$. Recall from Section 5.1 that the function g denotes a modular form of weight k and character $\bar{\chi}$ for $\Gamma(N)$. We start by decomposing the divisor $\operatorname{div}(g)$ on $X(N)$ as

$$\operatorname{div}(g) = \operatorname{div}(g)|_{X^0(N)} + \sum_{j=1}^{p_N} r_j P_j,$$

where the first summand corresponds to the part of the divisor having support on the open subset $X^0(N) \subseteq X(N)$ and the second summand is supported on the cusps. Now, consider the divisor C on $E(N)$ given by

$$C := p^* \operatorname{div}(g)|_{E^\circ(N)} + 2mH + \sum_{j=1}^{p_N} \sum_{v=0}^{N-1} \left(-mv + \frac{mN}{6} + r_j + \frac{m}{N} v^2 \right) \Theta_{j,v}. \tag{5.9}$$

Choose a smooth hermitian metric $||\cdot||'$ on the corresponding line bundle $\mathcal{O}_{E(N)}(C)$ and let s be a section of $\mathcal{O}_{E(N)}(C)$ such that $\operatorname{div}(s) = C$. The following proposition can be seen as a generalization of [BKK16, Proposition 4.9].

Proposition 5.2.9. *The hermitian line bundle $\overline{L}_{k,m,N}$ satisfies the following properties:*

(1) The restriction of the metric $||\cdot||_{\mathrm{Pet}}$ to $E^0(N)$ is smooth. Moreover, the divisor of the restriction of the function $G(\tau,z)$ to $E^0(N)$ is

$$p^* \operatorname{div}(g)|_{E^0(N)} + 2mH.$$

Therefore, there is a unique isomorphism

$$\alpha \colon L_{k,m,N} \simeq \mathcal{O}_{E^0(N)}(C)$$

sending $G(\tau,z)$ to s.

(2) Each point P belonging to one component $\Theta_{j,v}$ has a neighborhood U on which

$$\log ||G(\tau,z)||^2_{\mathrm{Pet}} = \log ||s||'^2 + \varphi_1,$$

where φ_1 is a pre-log-log function along B_{smooth}.

(3) On the affine coordinate chart $W^0_{j,v}$, we can write

$$\log ||G(\tau,z)||^2_{\mathrm{Pet}} = \log ||s||'^2 + \varphi_2 - \frac{m}{N} \frac{\log(u_v \bar{u}_v) \log(\nu_v \bar{\nu}_v)}{\log(u_v \bar{u}_v) + \log(\nu_v \bar{\nu}_v)},$$

where φ_2 is a pre-log-log function along B.

It follows that the 5-tuple $(1,\mathcal{O}_{E(N)}(C),T,||\cdot||,\alpha)$ is a Mumford–Lear extension of the hermitian line bundle $\overline{L}_{k,m,N}$ to $E(N)$ and that the divisor of $G(\tau,z)$ on the universal elliptic surface $E(N)$ is given by

$$\mathrm{div}_{E(N)}(G(\tau,z)) = C.$$

Proof. The proof of part (1) is essentially the same as in [BKK16, Proposition 4.9 (i)], noting that $\mathrm{div}(\eta)|_{E^0(N)} = 0$.

To prove part (2) and part (3) recall that we may work over the cusp P_1. Consider the open affine chart $W = W^0_{1,v}$. By Lemma 5.1.10 we have that

$$\begin{aligned}\log(||G(\tau,z)||^2_{\mathrm{Pet}})|_W =& \log(|G(\tau,z)|^2)|_W \\ &+\frac{m}{N}\left((v+1)^2\log(u_v\bar{u}_v)+v^2\log(v_v\bar{v}_v)-\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v)}\right)\\ &+k\log\left(-\frac{N}{4\pi}(\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v))\right).\end{aligned}$$

We study the summands of the above expression. For the term $\log|G(\tau,z)|^2$, we compute the multiplicity of $G(\tau,z)$ along $\Theta_{1,v}$: on the one hand, by the proof of [Kra91, Proposition 2.4], we have that the multiplicity of $\theta_{1,1}(\tau,z)$ at $\Theta_{1,v}$ is given by $(-v/2+N/8)$. On the other hand, we see that the multiplicity of $\eta(\tau)$ at $\Theta_{1,v}$ is given by $N/24$. Finally, we have that r_1 is the multiplicity of $g(\tau)$ at the cusp P_1. It follows that the multiplicity of $G(\tau,z)$ along $\Theta_{1,v}$ is given by $-mv+Nm/6+r_1$. Hence, on $W\setminus H$ we can write

$$\begin{aligned}\log\left(|G(\tau,z)|^2\right) =& \left(-m(v+1)+\frac{Nm}{6}+r_1\right)\log(u_v\bar{u}_v)\\ &+\left(-mv+\frac{Nm}{6}+r_1\right)\log(v_v\bar{v}_v)+\varphi_3\end{aligned}$$

where φ_3 is a smooth function. We next consider the remaining terms of the expression of $\log(||G(\tau,z)||^2_{\mathrm{Pet}})|_W$. By Lemma 5.2.8, the term

$$\frac{m}{N}\left(-\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v)}\right)$$

is pre-log-log along B_{smooth}. Also, we have that

$$k\log\left(-\frac{N}{4\pi}(\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v))\right)$$

is pre-log-log along B. Finally, the terms

$$\frac{mv^2}{N}\log(v_v\bar{v}_v) \quad \text{and} \quad \frac{m(v+1)^2}{N}\log(u_v\bar{u}_v)$$

add mv^2/N and $m(v+1)^2/N$ to the multiplicities of the components $\Theta_{1,v}$ and $\Theta_{1,v+1}$, respectively. Summing up, we obtain

$$\begin{aligned}\log(||G(\tau,z)||^2_{\mathrm{Pet}})|_W =& \left(-m(v+1)+\frac{Nm}{6}+r_1+\frac{m(v+1)^2}{N}\right)\log(u_v\bar{u}_v)\\ &+\left(-mv+\frac{Nm}{6}+r_1+\frac{mv^2}{N}\right)\log(v_v\bar{v}_v)\\ &-\frac{m}{N}\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v)}+\varphi_2\end{aligned}$$

where φ_2 is pre-log-log along B. This concludes the proof of the proposition. ■

By Proposition 5.2.9, we have that the Mumford–Lear extension $\overline{L}_{k,m,N}$ to $E(N)$ is isomorphic to $\mathcal{O}_{E(N)}(C)$. We now compute the self intersection product C^2.

Proposition 5.2.10. *The self intersection product C^2 is given by*

$$C^2 = \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3} + \frac{m^2\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3N^2}. \tag{5.10}$$

In particular, if $k = m = N = 4$, we have that $[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)] = 24$ and hence $C^2 = 136$, thus confirming the calculation in [BKK16, Proposition 4.10].

Proof. Recall that we have $\deg(p^*\mathrm{div}(g)) = k\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]/12$ and $p_N = [\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]/N$. We obtain

$$\begin{aligned}
C &= p^*\mathrm{div}(g)|_{E^\circ(N)} + 2mH + \sum_{j=1}^{p_N}\sum_{v=0}^{N-1}\left(-mv + \frac{mN}{6} + r_j + \frac{m}{N}v^2\right)\Theta_{j,v} \\
&= p^*\mathrm{div}(g) + 2mH + \sum_{j=1}^{p_N}\sum_{v=0}^{N-1}\left(-mv + \frac{mN}{6} + \frac{m}{N}v^2\right)\Theta_{j,v} \\
&\sim 2mH + \sum_{j=1}^{p_N}\sum_{v=0}^{N-1}\left(-mv + \frac{mN}{6} + \frac{\deg(p^*\mathrm{div}(g))}{p_N} + \frac{m}{N}v^2\right)\Theta_{j,v} \\
&\sim 2mH + \sum_{j=1}^{p_N}\sum_{v=0}^{N-1}\left(-mv + \frac{mN}{6} + \frac{kN}{12} + \frac{m}{N}v^2\right)\Theta_{j,v} =: C',
\end{aligned}$$

where $\sim$ denotes linear equivalence of divisors. Thus, $C^2 = (C')^2$. Moreover, it follows from the proof of [Kra91, Proposition 3.2] that

$$H \cdot H = -\frac{[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{12}.$$

Furthermore, we have that

$$H \cdot \Theta_{j,v} = \begin{cases} 1, & \text{if } v = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Theta_{j,v} \cdot \Theta_{j',v'} = \begin{cases} -2, & \text{if } v = v' \text{ and } j = j', \\ 1, & \text{if } j = j' \text{ and } v - v' \equiv \pm 1 \bmod N, \\ 0, & \text{otherwise.} \end{cases}$$

From these intersection products and the explicit description of C' we derive the result. ■

Theorem/Definition 5.2.11. *The hermitian line bundle of Jacobi forms $\overline{L}_{k,m,N}$ admits all Mumford–Lear extensions over the generalized elliptic curve $E(N)$. It thus defines a b-line bundle of Jacobi forms $\boldsymbol{\mathcal{L}}_{k,m,N} = \mathcal{O}_{E(N)}(\boldsymbol{D}_{k,m,N})$, where $\boldsymbol{D}_{k,m,N} = \left(D_{k,m,N,\Pi'}\right)_{\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}}$ is the b-divisor associated to the rational section $G(\tau,z)$. Moreover, $\boldsymbol{D}_{k,m,N}$ is of the form*

$$\boldsymbol{D}_{k,m,N} = p^*\mathrm{div}(g)|_{E^0(N)} + 2mH + \frac{m}{N}\boldsymbol{D}',$$

where $\boldsymbol{D}' = \left(D'_{\Pi'}\right)_{\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}}$ is the nef toroidal b-divisor given on each of the $N \cdot p_N$ 2-dimensional cones of $\Pi_{E(N)}$ by the concave rational function $\phi_{\boldsymbol{D}'}(a,b) = \frac{ab}{a+b}$.

Proof. Let P be a double point in T. Then the blow up at P is an allowable toroidal modification $E(N)_{\Pi'}$ of $E(N)$ corresponding to some conical complex $\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$. We denote by $\pi_{\Pi'} \colon E(N)_{\Pi'} \to E(N)$ the corresponding toroidal modification with exceptional divisor E. Then part (3) of Proposition 5.2.8 and part (3) of Proposition 5.2.9 imply that

$$\left(N, \pi_{\Pi'}^{*}\mathcal{O}_{E(N)}(NC) \otimes \mathcal{O}_{E(N)_{\Pi'}}\left(-\frac{m}{2}E\right)\right)$$

is a Mumford–Lear extension of $\overline{L}_{k,m,N}$ to $E(N)_{\Pi'}$. Moreover, we have that

$$\mathrm{div}_{E(N)_{\Pi'}}(G(\tau,z)) = \pi_{\Pi'}^{*}\mathrm{div}_{E(N)}(G(\tau,z)) - \frac{m}{2N}E.$$

The case of any birational model $E(N)_{\Pi'}$ for any $\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$ follows similarly. In particular, to any pair $\Pi'' \geq \Pi'$ in $R\left(\Pi_{E(N)}\right)$ such that the birational morphism $\pi \colon E(N)_{\Pi''} \to E(N)_{\Pi'}$ is a blow up at a double point P in the boundary divisor of $E(N)_{\Pi'}$ with exceptional divisor E', we can associate a pair of coprime integers (n', n'') (the so called *type* of P) such that the divisor $\mathrm{div}_{E(N)_{\Pi''}}(G(\tau,z))$ of $G(\tau,z)$ on $E(N)_{\Pi''}$ has the form

$$\mathrm{div}_{E(N)_{\Pi''}}(G(\tau,z)) = \pi^{*}\mathrm{div}_{E(N)_{\Pi'}}(G(\tau,z)) - \frac{m}{N}\frac{1}{n'n''(n'+n'')}E'. \tag{5.11}$$

The assignment of the pair of coprime integers to double points is done as follows: start with the point P in T and assign to it the pair $(1,1)$. Then blowing up P results in two double points P_1 and P_2 of types $(1,2)$ and $(2,1)$, respectively. We continue in this way in which a singular point P' in the singular locus of the boundary divisor $B_{\Pi'} = E(N)_{\Pi'} \setminus E^0(N)$ of type (n', n'') results in two double points P_1' and P_2' of multiplicities $(n', n'+n'')$ and $(n'+n'', n'')$, respectively. It follows from properties of the so called Stern–Brocot tree that this recipe covers all possible pairs of coprime integers.

The statement about the corresponding b-divisor follows from parts (1) and (3) of Proposition 5.2.9. ■

We now compute the degree of the b-divisor $\boldsymbol{D}_{k,m,N}$.

Theorem 5.2.12. *The b-divisor $\boldsymbol{D}_{k,m,N}$ is integrable and its self intersection number is given by*

$$\boldsymbol{D}_{k,m,N}^{2} = \frac{mk\left[\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)\right]}{3}.$$

Proof. We will give two proofs of this theorem.
Proof 1. Equation (5.11) implies that for any $\Pi'' \geq \Pi'$ in $R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$ corresponding to a blow up at a double point $P \in T$ of type (n', n''), we have that

$$D_{k,m,N,\Pi''}^{2} = D_{k,m,N,\Pi'}^{2} - \frac{m^2}{N^2}\frac{1}{n'^2 n''^2 (n'+n'')^2}. \tag{5.12}$$

Then, as in [BKK16, Theorem 4.11] and as in Example 1.4.7, we deduce the integrability of $\boldsymbol{D}_{k,m,N}$ from the absolute convergence of the series

$$\begin{aligned}\sum_{\substack{n',n''>0\\ (n',n'')=1}} \frac{1}{n'^2 n''^2 (n'+n'')^2} &= \left(\sum_{n',n''>0} \frac{1}{n'^2 n''^2 (n'+n'')^2}\right)\left(\sum_{n>0}\frac{1}{n^6}\right)^{-1} \\ &= \frac{\zeta(2,2;2)}{\zeta(6)} = \frac{\frac{1}{3}\zeta(6)}{\zeta(6)} = \frac{1}{3},\end{aligned}$$

where $\zeta(2,2;2)$ denotes the value of the so called *Mordell–Tornheim zeta function* at the triple $(2,2;2)$ and $\zeta(6)$ the value of the *Riemann zeta function* at 6. Indeed, using the fact that $E(N)$ has $N \cdot p_N = [\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]$ double points of type $(1,1)$, Equation (5.12) and Proposition 5.2.10 imply that

$$\begin{aligned}
\boldsymbol{D}^2_{k,m,N} &= \lim_{\Pi' \in R\left(\Pi_{E(N)}\right)} D^2_{k,m,N,\Pi'} \\
&= \lim_{\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}} D^2_{k,m,N,\Pi'} \\
&= C^2 - \frac{m^2 N p_N}{N^2} \sum_{\substack{n',n''>0 \\ (n',n'')=1}} \frac{1}{n'^2 n''^2 (n'+n'')^2} \\
&= \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3} + \frac{m^2\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3N^2} - \frac{m^2\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{N^2}\frac{1}{3} \\
&= \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3}.
\end{aligned}$$

We remark that this computation was first done in [BKK16, Theorem 4.11] for the case $k = m = 4$. It appears also in [KP15].

Proof 2. It follows from part (3) of Proposition 5.2.9 that the coefficient of the newly added exceptional divisors as we move along $R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$ is given in local coordinates by

$$-\frac{m}{N}\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v)}.$$

Setting $a := -\log(u_v\bar{u}_v)$ and $b := -\log(u_v\bar{u}_v)$ and dividing out by m/N gives the function

$$\phi_{\boldsymbol{D}'}(a,b) = \frac{ab}{a+b},$$

which is defined on the 2-dimensional cone corresponding to a small neighborhood of the double point $\Theta_v \cap \Theta_{v+1}$. Now, integrability of the b-divisor $\boldsymbol{D}_{k,m,N}$ follows from the nefness property of the toroidal b-divisor $\boldsymbol{D}'$ which in turn follows from the concavity of the function $\phi_{\boldsymbol{D}'}$. We compute the self intersection number $\boldsymbol{D}^2_{k,m,N}$ using Corollary 4.4.9 in conjunction with Proposition 3.4.22 and Example 3.5.28. We get

$$\begin{aligned}
\boldsymbol{D}^2_{k,m,N} &= \left(p^*\mathrm{div}(g)|_{E^0(N)} + 2mH + \frac{m}{N}\boldsymbol{D}'\right)^2 \\
&= D^2_{k,m,N,\Pi_{E(N)}} - \frac{m^2}{N^2}\left(D'^2_{\Pi_{E(N)}} - \boldsymbol{D}'^2\right) \\
&= C^2 - \frac{m^2}{N^2}\sum_{\sigma \in \Pi_{E(N)}(2)} c_\sigma \\
&= \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3} + \frac{m^2\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3N^2} - \frac{m^2}{N^2}\cdot N p_N \cdot \frac{1}{3} \\
&= \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3},
\end{aligned}$$

where c_σ is the correction term associated to any smooth 2-dimensional cone in $\Pi_{E(N)}$ given in Definition 3.5.23. This concludes the proof of the theorem. ■

Remark 5.2.13. We can rewrite the formula in the previous theorem as

$$\boldsymbol{D}_{k,m,N}^2 = \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{3} = mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]\,\frac{\zeta(2,2;2)}{\zeta(6)}.$$

Hence, this degree can be interpreted as the product of the weight of the Jacobi forms, their index, the index of the subgroup $\Gamma(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$ and a multiple zeta value. We refer to [KP15] and the references therein for an insight into the conjectural connection between multiple zeta values and arithmetic intersection numbers.

5.3 Interpretation

Throughout this section, N will denote an integer bigger than or equal to 3 and $\Gamma(N)$ will denote the principal congruence subgroup of level N. As before, we write $X(N)$ for the compactified modular curve of level N and $E(N)$ for the generalized universal elliptic curve of level N. Its associated conical complex will be denoted by $\Pi_{E(N)}$. In this section we will prove three statements which show why one should consider the whole tower of Mumford–Lear extensions of the line bundle of Jacobi forms, encoded in the b-divisor $\boldsymbol{D}_{k,m,N}$ on $E(N)$ instead of just the one extension $C = D_{k,m,N,\Pi_{E(N)}}$ to the canonical compactification $E(N)$. First, we show that there is an isomorphism between the space of global sections $H^0\left(E(N),\boldsymbol{D}_{k,m,N}\right)$ of the b-divisor of Jacobi forms and the space of Jacobi cusp forms $J_{k,m}^{\mathrm{cusp}}(\Gamma(N))$. Secondly, we give a Hilbert–Samuel formula relating the asymptotic growth of the dimension of the space of global sections of multiples of the b-divisor $\boldsymbol{D}_{k,m,N}$ with its degree. Finally we show that Chern–Weil theory holds in this context. The latter two results were shown in [BKK16, Theorem 5.1, Theorem 5.2] for the case $k = m = 4$, from which we cite some of the definitions and results of this section.

Geometric interpretation

Let X be a smooth projective variety of dimension n together with a snc divisor $B \subseteq X$. Let U be an open coordinate subset of X with coordinates $(z_1,\dots,z_n)$. We say that U *is adapted to* B, if the divisor B is locally given by the equation $z_1 \cdots z_k = 0$ for some $1 \le k \le n$.

For any conical complex $\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$, we denote by $B_{\Pi'}$ the boundary divisor given by $E(N)_{\Pi'} \setminus E^0(N)$ and by $T_{\Pi'}$ the set of double points in $B_{\Pi'}$.

The Mumford–Lear extensions $\mathcal{O}_{E(N)_{\Pi'}}(D_{k,m,N,\Pi'})$ of $\overline{L}_{k,m,N}$ to $E(N)_{\Pi'}$ for $\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$ from Proposition 5.2.9 and Theorem 5.2.12 were done by considering (non-trivial) sections s of $L_{k,m,N}$ of log-growth along $B_{\Pi'} \setminus T_{\Pi'}$. Explicitly, by considering sections s satisfying the following property: there exist a positive integer constant A such that the inequality

$$||s||^2 \le A(\log(u\bar{u}) + \log(v\bar{v}))^A$$

is satisfied along $B_{\Pi'} \setminus T_{\Pi'}$, i.e. for all local coordinates $(u,\, v)$ of a neighborhood of a point $P \in B_{\Pi'} \setminus T_{\Pi'}$ adapted to $B_{\Pi'}$. Hence, sections of the extended line bundle $\mathcal{O}_{E(N)_{\Pi'}}(D_{k,m,N,\Pi'})$ correspond to sections s of $L_{k,m,N}$ with growth $O((\log(u\bar{u}) + \log(v\bar{v}))^A)$ along $B_{\Pi'} \setminus T_{\Pi'}$, or equivalently, to sections s of $L_{k,m,N}$ satisfying

$$\lim_{|\log(u\bar{u})+\log(v\bar{v})|\to\infty} \frac{||s||^2}{|\log(u\bar{u}) + \log(v\bar{v})|^A} < \infty$$

along $B_{\Pi'} \setminus T_{\Pi'}$, for some positive integer constant A. We summarize these observations in the following lemma.

Lemma 5.3.1. *Let $\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$ and consider the corresponding allowable toroidal modification $E(N)_{\Pi'}$ of $E(N)$. Then the following conditions are equivalent:*

(1) $s \in H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right)$.

(2) There exist a positive integer constant A such that the inequality

$$||s||^2 \leq A|\log(u\bar{u}) + \log(v\bar{v})|^A$$

is satisfied along $B_{\Pi'} \setminus T_{\Pi'}$, i.e. for all local coordinates (u,v) of a neighborhood of a point $P \in B_{\Pi'} \setminus T_{\Pi'}$ adapted to $B_{\Pi'}$.

(3) There exists a positive integer constant A such that the inequality

$$\lim_{|\log(u\bar{u})+\log(v\bar{v})|\to\infty} \frac{||s||^2}{|\log(u\bar{u}) + \log(v\bar{v})|^A} < \infty$$

is satisfied along $B_{\Pi'} \setminus T_{\Pi'}$, i.e. for all local coordinates (u,v) of a neighborhood of a point $P \in B_{\Pi'} \setminus T_{\Pi'}$ adapted to $B_{\Pi'}$.

The following lemma relates the cusp condition of Jacobi forms (5.3) with a finiteness condition of the translation invariant metric $||\cdot||_{\mathrm{Pet}}$.

Lemma 5.3.2. *Let $f(\tau,z) \in J_{k,m}^{\mathrm{weak}}(\Gamma(N))$ be a weak Jacobi form of weight k and index m for $\Gamma(N)$. Then we have that*

$$f(\tau,z) \in J_{k,m}^{\mathrm{cusp}}(\Gamma(N)) \quad \textit{iff} \quad \lim_{\mathrm{Im}(\tau)\to\infty} \frac{||f(\tau,z)||_{\mathrm{Pet}}^2}{\mathrm{Im}(\tau)^k} < \infty.$$

Proof. Let $\tau = \xi + i\eta$ and $z = x + iy$. A Jacobi form $f(\tau,z) \in J_{k,m}^{\mathrm{weak}}(\Gamma(N))$ has a priori a Fourier expansion of the form

$$f(\tau,z) = \sum_{n\in\mathbb{N}, r\in\mathbb{Z}} c(n,r) q^n \zeta^r,$$

where $q = e^{2\pi i\tau/N}$, $\zeta = e^{2\pi i z}$. By the definition of the translation invariant metric, we have that

$$\begin{aligned}
||f(\tau,z)||_{\mathrm{Pet}}^2 &= |f(\tau,z)|^2 \exp\left(-4\pi m \frac{y^2}{\eta}\right)\eta^k \\
&= \sum_{\substack{n\in\mathbb{N}, r\in\mathbb{Z} \\ n'\in\mathbb{N}, r'\in\mathbb{Z}}} c(n,r)\bar{c}(n',r') q^n \bar{q}^{n'} \zeta^r \bar{\zeta}^{r'} \exp\left(-4\pi m \frac{y^2}{\eta}\right)\eta^k.
\end{aligned}$$

Then, by an averaging process, we get

$$\begin{aligned}
&\int_0^N \int_0^1 ||f(\tau,z)||_{\mathrm{Pet}}^2 \mathrm{d}x\,\mathrm{d}\xi = N \sum_{n\in\mathbb{N}, r\in\mathbb{Z}} |c(n,r)|^2 (q\bar{q})^n (\zeta\bar{\zeta})^r \exp\left(-4\pi m \frac{y^2}{\eta}\right)\eta^k \\
&= N \sum_{n\in\mathbb{N}, r\in\mathbb{Z}} |c(n,r)|^2 e^{2\pi i n(\tau-\bar{\tau})/N} e^{2\pi i r(z-\bar{z})} e^{-4\pi m y^2/\eta} \eta^k \\
&= N \sum_{n\in\mathbb{N}, r\in\mathbb{Z}} |c(n,r)|^2 e^{-4\pi(n\eta/N + ry + my^2/\eta)} \eta^k \\
&= N \sum_{n\in\mathbb{N}, r\in\mathbb{Z}} |c(n,r)|^2 e^{-4\pi N/\eta (n(\eta/N)^2 + r(\eta/N)y + (m/N)y^2)} \eta^k.
\end{aligned}$$

We see that in order to achieve convergence as $\eta \to \infty$ for all $y \in \mathbb{R}$, the quadratic form given by $n(\eta/N)^2 + r(\eta/N)y + (m/N)y^2$ has to be positive definite. Hence, we conclude that the following conditions are equivalent:

- $f(\tau,z) \in J^{\text{cusp}}_{k,m}(\Gamma(N))$
- for all $n \in \mathbb{N}$, $r \in \mathbb{Z}$, if $c(n,r) \neq 0$, then $n > 0$ and $4mn - Nr^2 > 0$
- for all $n \in \mathbb{N}$, $r \in \mathbb{Z}$, if $c(n,r) \neq 0$, then $\begin{pmatrix} n & r/2 \\ r/2 & m/N \end{pmatrix} > 0$
- for all $n \in \mathbb{N}$, $r \in \mathbb{Z}$, if $c(n,r) \neq 0$, then the quadratic form $n(\eta/N)^2 + r(\eta/N)y + (m/N)y^2$ is positive definite
- $$\lim_{\eta\to\infty} \frac{\int_0^N \int_0^1 ||f(\tau,z)||^2_{\text{Pet}} \mathrm{d}x\,\mathrm{d}\xi}{\eta^k} < \infty$$
- $$\lim_{\eta\to\infty} \frac{||f(\tau,z)||^2_{\text{Pet}}}{\eta^k} < \infty,$$

like we wanted to show. ■

We recall the definition of the space of global sections $H^0(E(N),\boldsymbol{D}_{k,m,N})$ of the b-divisor $\boldsymbol{D}_{k,m,N}$ given in Section 1.5.

$$\begin{aligned} H^0\left(E(N),\boldsymbol{D}_{k,m,N}\right) &:= \{f \in \mathbb{C}(E(N)) \mid b\text{-div}(f) + \lfloor \boldsymbol{D}_{k,m,N} \rfloor \geq 0\} \cup \{0\} \\ &= \bigcap_{\Pi' \in R\left(\Pi_{E(N)}\right)_{\text{snc}}} H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right). \end{aligned} \tag{5.13}$$

The following theorem gives a geometric interpretation of Jacobi cusp forms.

Theorem 5.3.3. *Let notations be as above. There is an isomorphism*

$$J^{\text{cusp}}_{k,m}(\Gamma(N)) \simeq H^0\left(E(N),\boldsymbol{D}_{k,m,N}\right)$$

between the space of global sections of the b-divisor $\boldsymbol{D}_{k,m,N}$ and the space of Jacobi cusp forms of weight k and index m for $\Gamma(N)$.

Proof. Let (u_ν, v_ν) be the local coordinates around $\Theta_\nu \cap \Theta_{\nu+1}$ as defined in Section 5.1. Moreover, let $f(\tau,z) \in J^{\text{cusp}}_{k,m}(\Gamma(N))$ be a Jacobi cusp form and let Π' be any conical complex in $R\left(\Pi_{E(N)}\right)_{\text{snc}}$. By Equation (5.13) above, to show that $f(\tau,z)$ is in $H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right)$, it suffices to show that $f(\tau,z)$ satisfies one of the equivalent conditions of Lemma 5.3.1. In order to do this, we may assume that $\pi_{\Pi'}\colon E(N)_{\Pi'} \to E(N)$ is a blow up at the double point $\Theta_\nu \cap \Theta_{\nu+1}$. Hence, it is given in local coordinates by $u_\nu = u^a v^b$ and $v_\nu = u^c v^d$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. In these coordinates η is of the form

$$\eta = -\frac{N}{4\pi}\left((a+c)\log(u\bar{u}) + (b+d)\log(v\bar{v})\right).$$

We assume that $|u|, |v| < 1$. Now, since $f(\tau,z) \in J^{\text{cusp}}_{k,m}(\Gamma(N))$, by Lemma 5.3.2 we have that

$$\lim_{\eta\to\infty} \frac{||f(\tau,z)||^2_{\text{Pet}}}{\eta^k} = \lim_{|(a+c)\log(u\bar{u})+(b+d)\log(v\bar{v})|\to\infty} \left(\frac{4\pi}{N}\right)^k \frac{||f(\tau,z)||^2_{\text{Pet}}}{\left((a+c)\log(u\bar{u}) + (b+d)\log(v\bar{v})\right)^k} < \infty.$$

With a suitable implied positive constant we have

$$\frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{|\log(u\bar{u})+\log(v\bar{v})|^k} \ll \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{((a+c)\log(u\bar{u})+(b+d)\log(v\bar{v}))^k}$$

from which we conclude that

$$\begin{aligned}
&\lim_{|\log(u\bar{u})+\log(v\bar{v})|\to\infty} \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{|\log(u\bar{u})+\log(v\bar{v})|^k} \\
&\ll \lim_{|(a+c)\log(u\bar{u})+(b+d)\log(v\bar{v})|\to\infty} \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{((a+c)\log(u\bar{u})+(b+d)\log(v\bar{v}))^k} \\
&= \lim_{\eta\to\infty}\left(\frac{N}{4\pi}\right)^k \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{\eta^k} < \infty.
\end{aligned}$$

From this we derive that $f \in H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right)$.

Conversely, let $f \in H^0(E(N),\boldsymbol{D}_{k,m,N})$. By Equation (5.13) and Lemma 5.3.1, for every blow up $\pi_{\Pi'}\colon E(N)_{\Pi'} \to E(N)$ there exist a non-negative integer $A_{\Pi'}$ such that the inequality

$$\lim_{|\log(u\bar{u})+\log(v\bar{v})|\to\infty} \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{|\log(u\bar{u})+\log(v\bar{v})|^{A_{\Pi'}}} < \infty$$

is satisfied for any choice of local coordinates $(u,\, v)$ of any neighborhood of a point $P \in B_{\Pi'} \setminus T_{\Pi'}$ adapted to $B_{\Pi'}$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ such that with a suitable implied constant we have that

$$\frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{|\log(u\bar{u})+\log(v\bar{v})|^{A_{\Pi_M}}} \gg \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{((a+c)\log(u\bar{u})+(b+d)\log(v\bar{v}))^k}$$

is satisfied for the choice of local coordinates (u,v) as above, where $E(N)_{\Pi_M}$ denotes the blow up given in local coordinates by the entries in M. Thus, we obtain

$$\begin{aligned}
\lim_{\eta\to\infty}\left(\frac{N}{4\pi}\right)^k \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{\eta^k} &= \lim_{|(a+c)\log(u\bar{u})+(b+d)\log(v\bar{v})|\to\infty} \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{((a+c)\log(u\bar{u})+(b+d)\log(v\bar{v}))^k} \\
&\ll \lim_{|\log(u\bar{u})+\log(v\bar{v})|\to\infty} \frac{||f(\tau,z)||^2_{\mathrm{Pet}}}{|\log(u\bar{u})+\log(v\bar{v})|^{A_{\Pi_M}}} < \infty.
\end{aligned}$$

The theorem now follows from Lemma 5.3.2. ■

Hilbert–Samuel formula

Using Theorem 5.3.3, we will show that the b-divisor $\boldsymbol{D}_{k,m,N}$ of Jacobi forms satisfies a Hilbert–Samuel type formula. We start with the following Lemma.

Lemma 5.3.4. *Let notations be as above. The limit (in terms of nets)*

$$\lim_{\Pi' \in R\left(\Pi_{E(N)}\right)} \dim H^0\left(E(N), D_{k,m,N,\Pi'}\right)$$

is given by

$$\frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{6} - \frac{m\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{8} + \frac{m\,[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{N} + \frac{[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)]}{24}.$$

Proof. The statement in [Kra91, Proposition 3.2] extends to show that for any $\Pi' \in R\left(\Pi_{E(N)}\right)_{\mathrm{snc}}$, the divisor $D_{k,m,N,\Pi_{E(N)}} - K_{E(N)_{\Pi'}}$ is ample. Here, $K_{E(N)_{\Pi'}}$ denotes the canonical divisor of $E(N)_{\Pi'}$. Hence, using the Kodaira Vanishing Theorem and the Riemann–Roch Theorem we get

$$\dim H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right) = \frac{1}{2}\left(D^2_{k,m,N,\Pi'} - D_{k,m,N,\Pi'} \cdot K_{E(N)_{\Pi'}}\right) + g_A\left(E(N)_{\Pi'}\right) + 1,$$

where $g_A\left(E(N)_{\Pi'}\right) = g_A\left(E(N)\right) = \left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]/24 - 1$ is the arithmetic genus of the surface $E(N)_{\Pi'}$. Moreover, the calculations carried out in the proof of [Kra91, Proposition 3.2] show that

$$\begin{aligned} D_{k,m,N,\Pi'} \cdot K_{E(N)_{\Pi'}} &= D_{k,m,N,\Pi_{E(N)}} \cdot K_{E(N)} = C \cdot K_{E(N)} \\ &= 2mH \cdot K_{E(N)} = \frac{m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{4} - \frac{2m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{N}, \end{aligned}$$

where H and C are the divisors given in Equation (5.9). Summing up, we obtain

$$\begin{aligned} &\dim H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right) \\ &= \frac{1}{2}\left(D^2_{k,m,N,\Pi'} - \frac{m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{4} + \frac{2m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{N}\right) + \frac{\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{24}. \end{aligned}$$

Finally, taking the limit, we get

$$\begin{aligned} &\lim_{\Pi' \in R\left(\Pi_{E(N)}\right)} \dim H^0\left(E(N)_{\Pi'}, D_{k,m,N,\Pi'}\right) \\ &= \frac{1}{2}\left(\boldsymbol{D}^2_{k,m,N} - \frac{m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{4} + \frac{2m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{N}\right) + \frac{\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{24} \\ &= \frac{mk\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{6} - \frac{m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{8} + \frac{m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{N} + \frac{\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{24}, \end{aligned}$$

thus proving the statement of the lemma. ■

We are now ready to prove a Hilbert–Samuel type formula.

Corollary 5.3.5. *The b-divisor $\boldsymbol{D}_{k,m,N}$ satisfies the Hilbert–Samuel type formula*

$$\boldsymbol{D}^2_{k,m,N} = \lim_{\ell\to\infty} \frac{\dim J^{\mathrm{cusp}}_{\ell k,\ell m}(\Gamma(N))}{\ell^2/2!} = \lim_{\ell\to\infty} \frac{\dim H^0\left(E(N), \ell\boldsymbol{D}_{k,m,N}\right)}{\ell^2/2!}.$$

Proof. Note that by nefness, the dimension of the spaces $H^0\left(E(N), D_{k,m,N,\Pi'}\right)$ for $\Pi' \in R\left(\Pi_{E(N)}\right)$ is monotone decreasing. Hence, we have that

$$\lim_{\ell\to\infty} \dim H^0\left(E(N), \ell\boldsymbol{D}_{k,m,N}\right) = \lim_{\ell\to\infty} \lim_{\Pi' \in R\left(\Pi_{E(N)}\right)} \dim H^0\left(E(N)_{\Pi'}, \ell D_{k,m,N,\Pi'}\right).$$

Therefore, by Lemma 5.3.4 we get

$$\begin{aligned} &\lim_{\ell\to\infty} \frac{\dim H^0\left(E(N), \ell\boldsymbol{D}_{k,m,N}\right)}{\ell^2/2} = \frac{\lim_{\ell\to\infty} \lim_{\Pi' \in R\left(\Pi_{E(N)}\right)} \dim H^0\left(E(N)_{\Pi'}, \ell D_{k,m,N,\Pi'}\right)}{\ell^2/2} = \\ &\lim_{\ell\to\infty} \left(\frac{\frac{\ell^2 mk\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{6} - \frac{\ell m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{8} + \frac{\ell m\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{N} + \frac{\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{24}}{\frac{\ell^2}{2}}\right) \\ &= \frac{mk\left[\mathrm{PSL}_2(\mathbb{Z})\colon \Gamma(N)\right]}{3} = \boldsymbol{D}^2_{k,m,N}, \end{aligned}$$

concluding the proof of the corollary. ■

This formula reflects the fact that the translation invariant metric encodes the asymptotic behavior of the space of Jacobi cusp forms of weight k and index m for $\Gamma(N)$.

Remark 5.3.6. Note that by Remark 5.1.7 and by Theorem 5.2.12 we get immediately the Hilbert–Samuel equation

$$\begin{aligned}\lim_{\ell\to\infty}\frac{\dim J_{\ell k,\ell m}(\Gamma(N))}{\ell^2/2!} &= \lim_{\ell\to\infty}\left(\frac{\ell^2 mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon\Gamma(N)]}{6}+o(\ell^2)\right)\left(\frac{\ell^2}{2!}\right)^{-1}\\ &= \frac{mk\,[\mathrm{PSL}_2(\mathbb{Z})\colon\Gamma(N)]}{3}=\boldsymbol{D}^2_{k,m,N}.\end{aligned}$$

However, the proof of Corollary 5.3.5 using Theorem 5.3.3 gives a more conceptual argument for why such an equality holds true.

5.3.1 Chern–Weil theory for b-divisors

We show that the self intersection product in the sense of b-divisors is compatible with Chern–Weil theory in the case of the b-line bundle of Jacobi forms. This is a straightforward generalization of [BKK16, Theorem 5.2].

We write $c_1(L_{k,m,N},||\cdot||_{\mathrm{Pet}})=\frac{1}{2\pi i}\partial\bar{\partial}\log||G(\tau,z)||^2_{\mathrm{Pet}}$ for the first Chern class of the hermitian line bundle $\overline{L}_{k,m,N}$.

Theorem 5.3.7. *With notations as above, the equality*

$$\boldsymbol{D}^2_{k,m,N}=\int_{E(N)}c_1(L_{k,m,N},||\cdot||_{\mathrm{Pet}})^{\wedge 2}$$

holds true.

Proof. The proof will follow the same lines as the proof of [BKK16, Theorem 5.2]. By Propositions 5.2.8 and 5.2.9, we know that the integral on the right hand side exists and is finite. Now, let $C\in\mathrm{Div}(E(N))_{\mathbb{Q}}$ be the divisor in Equation (5.9) and choose a pre-log metric $||\cdot||'$ on $\mathcal{O}_{E(N)}(C)$ such that each double point $P_{j,v}=\Theta_{j,v}\cap\Theta_{j,v+1}\in W^0_{j,v}$ has a neighborhood in which $\log||G(\tau,z)||^2_{\mathrm{Pet}}$ is given by

$$\log||G(\tau,z)||^2_{\mathrm{Pet}}=\log||G(\tau,z)||'^2-\frac{m}{N}\frac{\log(u_v\bar{u}_v)\log(v_v\bar{v}_v)}{\log(u_v\bar{u}_v)+\log(v_v\bar{v}_v)}. \tag{5.14}$$

In order to simplify notation, let us write $\omega:=c_1(L_{k,m,N},||\cdot||_{\mathrm{Pet}})$, $\omega':=c_1(L_{k,m,N},||\cdot||')$ and $f:=\log||G(\tau,z)||^2_{\mathrm{Pet}}-\log||G(\tau,z)||'^2$. Note that

$$\omega=\omega'+\frac{1}{2\pi i}\partial\bar{\partial}f$$

holds true. Moreover, since Chern–Weil theory can be extended to pre-log singularities, we have that

$$\int_{E(N)}\omega'^{\wedge 2}=C^2$$

is satisfied. Hence, we obtain

$$\begin{aligned}\int_{E(N)}\omega^{\wedge 2}&=\int_{E(N)}\omega'^{\wedge 2}-\int_{E(N)}\mathrm{d}\left(\frac{2}{2\pi i}\partial f\wedge\omega'+\frac{1}{(2\pi i)^2}\partial f\wedge\partial\bar{\partial}f\right)\\ &=C^2-\int_{E(N)}\mathrm{d}\left(\frac{2}{2\pi i}\partial f\wedge\omega'+\frac{1}{(2\pi i)^2}\partial f\wedge\partial\bar{\partial}f\right).\end{aligned}$$

We are thus led to compute the integral on the right hand side. As can be seen in [BKK16, Theorem 5.2], to compute this integral it suffices to compute the residues at the double points $P_{j,\nu}$. To compute these, for each point $P_{j,\nu}$ and for $0 < \varepsilon < 1/e$, let $V_{j,\nu,\varepsilon}$ be the poly-cylinder defined by

$$V_{j,\nu,\varepsilon} = \left\{ (u_\nu, v_\nu) \in W^0_{j,\nu} \,|\, |u_\nu| \le \varepsilon,\, |v_\nu| \le \varepsilon \right\}.$$

We write (u, v) for the coordinates (u_ν, v_ν). We decompose the boundary $\partial V_{j,\nu,\varepsilon}$ into $A_\varepsilon \cup B_\varepsilon$, where

$$\begin{aligned} A_\varepsilon &= \left\{ (u,v) \in W^0_{j,\nu} \,|\, |u| \le \varepsilon,\, |v| = \varepsilon \right\}, \\ B_\varepsilon &= \left\{ (u,v) \in W^0_{j,\nu} \,|\, |u| = \varepsilon,\, |v| \le \varepsilon \right\}. \end{aligned}$$

Now, using equation 5.14 together with the Stokes Theorem, we have that

$$\begin{aligned} \int_{A_\varepsilon} \frac{1}{(2\pi i)^2} \partial f \wedge \partial\bar{\partial} f &= \frac{m^2}{N^2} \int_0^\varepsilon \frac{2(\log(\varepsilon^2))^2 \log(r^2) 2r}{(\log(r^2) + \log(\varepsilon^2))^4 r^2} \mathrm{d}r = -\frac{m^2}{6N^2}, \\ \int_{B_\varepsilon} \frac{1}{(2\pi i)^2} \partial f \wedge \partial\bar{\partial} f &= \frac{m^2}{N^2} \int_0^\varepsilon \frac{2(\log(\varepsilon^2))^2 \log(r^2) 2r}{(\log(r^2) + \log(\varepsilon^2))^4 r^2} \mathrm{d}r = -\frac{m^2}{6N^2}. \end{aligned}$$

Hence, we get that

$$\lim_{\varepsilon \to 0} \int_{\partial V_{j,\nu,\varepsilon}} \left(\frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial\bar{\partial} f \right) = -\frac{m^2}{3N^2}.$$

Therefore, the sequence of equalities

$$\begin{aligned} &- \int_{E(N)} \mathrm{d} \left(\frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial\bar{\partial} f \right) \\ &= \sum_{j=1}^{p_N} \sum_{\nu=0}^{N-1} \lim_{\varepsilon \to 0} \int_{\partial V_{j,\nu,\varepsilon}} \left(\frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial\bar{\partial} f \right) \\ &= N p_N \frac{-m^2}{3N^2} = -\frac{m^2 \, [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]}{3N^2} \end{aligned}$$

is satisfied. We conclude that

$$\begin{aligned} \int_{E(N)} c_1(L_{k,m,N}, ||\cdot||_{\mathrm{Pet}})^{\wedge 2} &= \int_{E(N)} \omega^{\wedge 2} \\ &= \int_{E(N)} \omega'^{\wedge 2} - \int_{E(N)} \mathrm{d} \left(\frac{2}{2\pi i} \partial f \wedge \omega' + \frac{1}{(2\pi i)^2} \partial f \wedge \partial\bar{\partial} f \right) \\ &= C^2 - \frac{m^2 \, [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]}{3N^2} \\ &= \frac{mk \, [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]}{3} + \frac{m^2 \, [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]}{3N^2} - \frac{m^2 \, [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]}{3N^2} \\ &= \frac{mk \, [\mathrm{PSL}_2(\mathbb{Z}) \colon \Gamma(N)]}{3} = \boldsymbol{D}^2_{k,m,N} \end{aligned}$$

holds true, as we wanted to show. ∎

Let $C' \subseteq E(N)$ be any curve not contained in the boundary $E(N) \setminus E^0(N)$. Recall that we may view C' as a Cartier b-divisor $\boldsymbol{C'} = \left(C'_{\Pi'}\right)_{\Pi' \in R\left(\Pi_{E(N)}\right)}$ by taking $C'_{\Pi'} = \pi^*_{\Pi'} C'$ for every $\Pi' \in R\left(\Pi_{E(N)}\right)$. It turns out that we can compute the top intersection number of $\boldsymbol{D}_{k,m,N}$ with $\boldsymbol{C'}$ as an integral over C' using the differential form $c_1(L_{k,m,N}, ||\cdot||_{\mathrm{Pet}})$. The proof of the following theorem is analogous to the one in [BKK16, Theorem 5.5].

Theorem 5.3.8. *Let notations be as above. The top intersection number $\boldsymbol{D}_{k,m,N} \cdot \boldsymbol{C'}$ of b-divisors is given by*

$$\boldsymbol{D}_{k,m,N} \cdot \boldsymbol{C'} = \int_{C'} c_1(L_{m,k,N}, ||\cdot||_{\mathrm{Pet}}).$$

Appendix A

Integrability of (not necessarily nef) toric b-divisors

Throughout this appendix, $\Sigma \subseteq N_{\mathbb{R}}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. We give a starting point to study integrability questions of (not necessarily nef) toric b-divisors on the toric variety X_Σ. In order to do this we state and give an example concerning a result in [HKP06] which expresses the self intersection number of a (not necessarily nef) toric Cartier divisor as a linear combination of volumes of bounded regions of a hyperplane arrangement.

We start by introducing some definitions and notations.

Definition A.1. The *Euler characteristic of a fan* Σ is defined by

$$\epsilon(\Sigma) := \sum_{j=0}^{n} (-1)^j \# \Sigma(j).$$

For each $I \subseteq \Sigma(1)$ let Σ_I be the fan consisting of exactly those cones in Σ spanned by rays in I. Also, if $D = \sum_{\tau \in \Sigma(1)} a_\tau D_\tau$ is a toric Weil divisor, we define the polyhedral region

$$P_{D,I} := \{u \in M_{\mathbb{R}} \,\big|\, \langle u, v_\tau \rangle \geq -a_\tau \text{ if and only if } \tau \in I\}.$$

While I ranges through all subsets of $\Sigma(1)$, these polyhedral regions give a decomposition of $M_{\mathbb{R}}$, i.e. a hyperplane arrangement in $M_{\mathbb{R}}$.

The following theorem expresses the self intersection number of a toric Cartier divisor as a linear combination of volumes of bounded regions of the hyperplane arrangement given above. The proof can be found in [HKP06, Theorem 1].

Theorem A.2. *Let notations be as above. Then we have that*

$$D^n = (-1)^n \sum_{P_{D,I} \text{bdd}} \epsilon(\Sigma_I) \operatorname{vol}(P_{D,I}).$$

Now, let $\Sigma'' \geq \Sigma' \in R(\Sigma)$ and suppose that $\Sigma''(1) = \Sigma'(1) \cup \{\tau\}$. Note that for $K \subseteq \Sigma'(1)$ we have

$$P_{D_{\Sigma'},K} = P_{D_{\Sigma''},K} \cup P_{D_{\Sigma''},K \cup \{\tau\}}. \tag{A.1}$$

In particular, if $K \subseteq \Sigma'(1)$ is such that $P_{D_{\Sigma'},K}$ is bounded, then so are both $P_{D_{\Sigma''},K}$ and $P_{D_{\Sigma''},K\cup\{\tau\}}$.

Note however that there are other subsets $J \subseteq \Sigma''(1)$ not coming from subsets of $\Sigma'(1)$ for which $P_{D_{\Sigma''},J}$ is bounded. Let S denote the subset of those J's.

We have

$$
\begin{aligned}
(-1)^n D_{\Sigma''}^n &= \sum_{P_{D_{\Sigma''},I}} \epsilon(\Sigma''_I)\mathrm{vol}\left(P_{D_{\Sigma''},I}\right) \\
&= \sum_{P_{D_{\Sigma''},K}} \epsilon(\Sigma''_K)\mathrm{vol}\left(P_{D_{\Sigma''},K}\right) + (-1)^n \sum_{P_{D_{\Sigma''},K\cup\{\tau\}}} \epsilon(\Sigma''_{K\cup\{\tau\}})\mathrm{vol}\left(P_{D_{\Sigma''},K\cup\{\tau\}}\right) \\
&\quad + \sum_{P_{D_{\Sigma''},J}} \epsilon(\Sigma''_J)\mathrm{vol}\left(P_{D_{\Sigma''},J}\right),
\end{aligned}
$$

where I ranges through all subsets of $\Sigma''(1)$ with bounded $P_{D_{\Sigma''},I}$, K ranges through all subsets of $\Sigma'(1)$ with bounded $P_{D_{\Sigma'},K}$ and J ranges through S.

Ideally, we would like to express the right hand side of the above equation as $(-1)^n D_{\Sigma'}^n$ plus some extra factor, so as to have an explicit formula for the correction term $D_{\Sigma''}^n - D_{\Sigma'}^n$.

We now give an example which illustrates the formula in Theorem A.2.

Example A.3. Consider the Hirzebruch surface $\mathcal{H}_2$. This is the toric surface given by the fan Σ in Figure A.1.

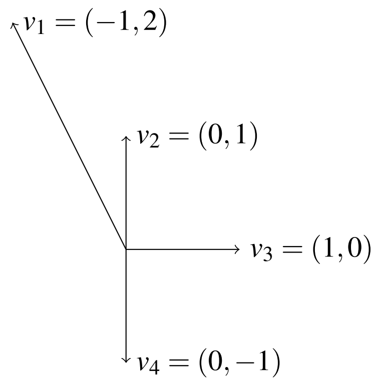


Figure A.1: Fan of $\mathcal{H}_2$

To simplify notation, we denote by i the ray spanned by the primitive vector v_i and the corresponding toric divisor by D_i. Consider the toric divisor $D_\Sigma = D_2 + D_4$. This toric divisor is not nef. This can be easily seen, as its corresponding support function is not concave. Figure A.2 shows the hyperplane arrangement corresponding to subsets $I \subseteq \{1,2,3,4\}$.

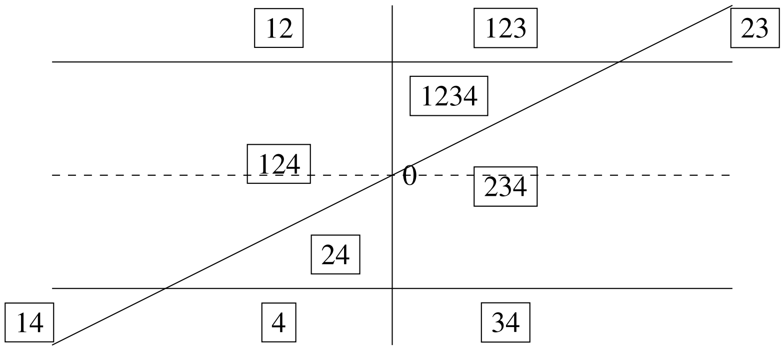


Figure A.2: Hyperplane arrangement corresponding to D_Σ

Note that the only bounded regions are the ones corresponding to $I_1 = \{1,2,3,4\}$ and $I_2 = \{2,4\}$. We also have $\epsilon(\Sigma_{I_1}) = 1$ and $\epsilon(\Sigma_{I_2}) = -1$. Hence, we get

$$D_\Sigma^2 = 2\,\mathrm{vol}\,(P_{D_\Sigma,I_1}) - 2\,\mathrm{vol}\,(P_{D_\Sigma,I_2}) = 0.$$

Consider now the fan $\Sigma' \geq \Sigma$ obtained by inserting the ray spanned by the primitive vector $v_5 = (1,1)$ as in Figure A.3.

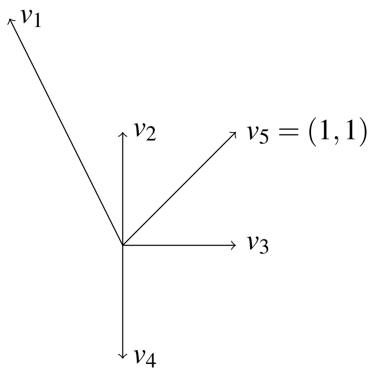


Figure A.3: Fan of blow up of $\mathcal{H}_2$

On $X_{\Sigma'}$ consider the toric divisor $D_{\Sigma'} = D_2 + D_4 - \frac{1}{2}D_5$. Note that we have the inequality

$$\psi_{D_{\Sigma'}}(v_5) = \frac{1}{2} \leq 1 = \psi_{D_{\Sigma'}}(v_2) + \psi_{D_{\Sigma'}}(v_3),$$

thus, we are not in the nef case. The corresponding hyperplane arrangement can be seen in Figure A.4.

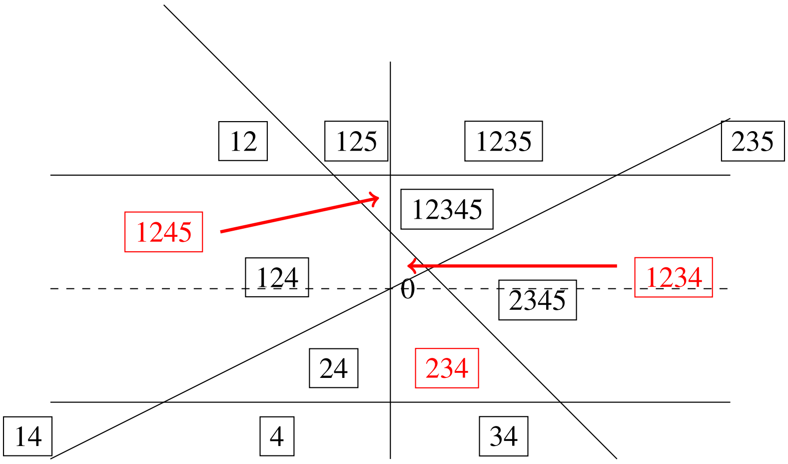


Figure A.4: Hyperplane arrangement corresponding to $D_{\Sigma'}$

Here we have

$$\epsilon\left(\Sigma'_{\{12345\}}\right) = 1,\, \epsilon\left(\Sigma'_{\{24\}}\right) = -1,\, \epsilon\left(\Sigma'_{\{1245\}}\right) = 0,\, \epsilon\left(\Sigma'_{\{234\}}\right) = -1,\, \epsilon\left(\Sigma'_{\{1234\}}\right) = 0.$$

Hence,

$$\begin{aligned} D^2_{\Sigma'} &= 2\left(\mathrm{vol}\left(P_{D_{\Sigma'},\{12345\}}\right) - \mathrm{vol}\left(P_{D_{\Sigma'},\{24\}}\right) - \mathrm{vol}\left(P_{D_{\Sigma'},\{234\}}\right)\right) \\ &= 2\left(\int_0^{1/3}\left(\frac{1}{2}+x\right)\mathrm{d}x + \int_{1/3}^{2}\left(1-\frac{x}{2}\right)\mathrm{d}x - 1 - \int_0^{1/3}\left(\frac{x}{2}+1\right)\mathrm{d}x - \int_{1/3}^{3/2}\left(\frac{3}{2}-x\right)\mathrm{d}x\right) \\ &= 2\cdot\frac{11}{12} - 2 - \frac{25}{12} = -\frac{9}{4}. \end{aligned}$$

We have

$$D^2_{\Sigma'} - D^2_{\Sigma} = -\frac{9}{4}.$$

Let us do a further blow up. Consider the fan Σ'' as shown in the Figure A.5.

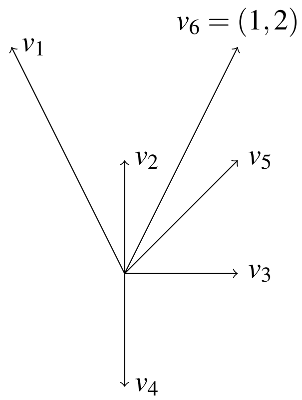


Figure A.5: Fan of a further blow up of $\mathcal{H}_2$

On $X_{\Sigma''}$, consider the toric divisor $D_{\Sigma''} = D_2 + D_4 - \frac{1}{2}D_5 - 3D_6$. Here, we have the inequality

$$\psi_{D_{\Sigma''}}(v_6) = 3 \geq \frac{5}{2} = \psi_{D_{\Sigma''}}(v_2) + \psi_{D_{\Sigma''}}(v_5).$$

Hence, we are still in the non-nef case. The corresponding hyperplane arrangement can be seen in Figure A.6, where we have only labelled the bounded regions.

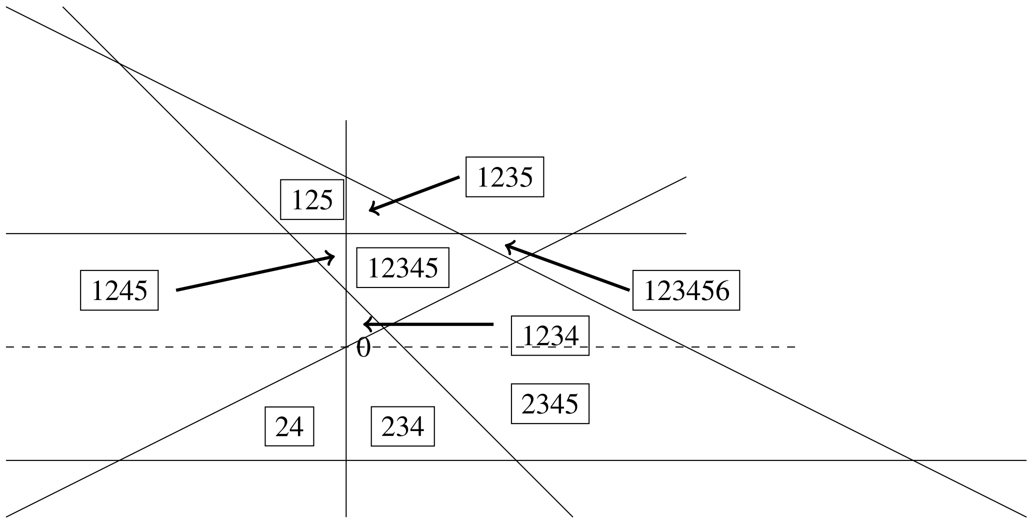


Figure A.6: Hyperplane arrangement corresponding to $D_{\Sigma''}$

Hence, we get

$$\epsilon\left(\Sigma''_{\{123456\}}\right) = 1,\ \epsilon\left(\Sigma''_{\{12345\}}\right) = 0,\ \epsilon\left(\Sigma''_{\{125\}}\right) = -1,\ \epsilon\left(\Sigma''_{\{1245\}}\right) = -1,\ \epsilon\left(\Sigma''_{\{24\}}\right) = -1,$$
$$\epsilon\left(\Sigma''_{\{1235\}}\right) = -1,\ \epsilon\left(\Sigma''_{\{1234\}}\right) = 0,\ \epsilon\left(\Sigma'_{\{234\}}\right) = -1,\ \epsilon\left(\Sigma''_{\{2345\}}\right) = -1.$$

We compute

$$\begin{aligned} D_{\Sigma''}^2 &= 2\left(\operatorname{vol}\left(P_{D_{\Sigma''},\{123456\}}\right) - \operatorname{vol}\left(P_{D_{\Sigma''},\{125\}}\right) - \operatorname{vol}\left(P_{D_{\Sigma''},\{1245\}}\right) - \operatorname{vol}\left(P_{D_{\Sigma''},\{24\}}\right)\right) \\ &\quad - 2\left(\operatorname{vol}\left(P_{D_{\Sigma''},\{1235\}}\right) - \operatorname{vol}\left(P_{D_{\Sigma''},\{2345\}}\right) - \operatorname{vol}\left(P_{D_{\Sigma''},\{234\}}\right)\right) \\ &= -\frac{29}{2}. \end{aligned}$$

We have

$$D_{\Sigma''}^2 - D_{\Sigma'}^2 = -\frac{29}{2} + \frac{9}{4} = -\frac{49}{4}.$$

Remark A.4. As this 2-dimensional example shows, it appears not to be so easy to come up with a formula for the degree of a non-nef, integrable toric b-divisor in terms of volumes of bounded regions in the plane. However, it is an approach which would work in arbitrary dimensions. To be able to do this, we need to solve the following combinatorial problem: suppose we are given a hyperplane arrangement H_0 in $\mathbb{R}^n$ and a function $\phi\colon \mathbb{R}^n \to \mathbb{Q}$. Choose any enumeration on the set of primitive integer vectors $v_1, v_2, \ldots$. We then recursively form the hyperplane arrangements $H_{i+1} = H_i \cup \ell_{i+1}$ where ℓ_{i+1} is the line defined by $\langle v_i + 1, x\rangle = -\phi(v_{i+1})$. We would like to know how the sum of the volumes of the bounded regions increase as we let this process go to infinity, i.e. if we let $i \to \infty$. We would also like to have a convergence criterion in terms of the function ϕ.

Appendix B

Some questions

Throughout this appendix, $\Sigma \subseteq N_{\mathbb{R}}$ will denote a complete and smooth fan of dimension n, i.e. such that $\dim(X_\Sigma) = n$. We formulate two questions. The first one concerns a possible relationship between toric metrics on a toric line bundle and toric b-divisors. This relationship could lead to the definition of the *height* of a toric b-line bundle and would have some interesting arithmetic applications. The second question concerns the Proj of the graded algebra coming from the global sections of multiples of a b-divisor for which we give some partial answers.

On toric metrics on toric b-line bundles

Given a big line bundle L on a smooth projective complex variety X, there is a way to associate a convex function $\hat{\psi}$ on the Okounkov body Δ_L defined by L to any continuous metric ψ on L. Moreover, this association satisfies

$$\deg_X(L, \psi) = \int_{\mathrm{int}(\Delta_L)} \hat{\psi}\, \mathrm{d}\lambda,$$

where $\mathrm{d}\lambda$ is the Lebesgue measure on $\mathrm{int}(\Delta_L)$ and $\deg_X(L,\psi)$ is the so called *metric degree*. We refer to ([Nys14]) for definitions and a proof of this fact.

Now, assume that we have a big and nef toric b-divisor $\boldsymbol{D}$ together with a concave function ψ on $\mathrm{int}(\Delta_{\boldsymbol{D}})$. Consider the Legendre dual $\psi^\vee$ defined on $N_{\mathbb{R}}$ (taking possibly $-\infty$ values). For any fan $\Sigma' \in R(\Sigma)$ we may consider the piecewise linear function $\psi^\vee_{\Sigma'}$ induced by $\psi^\vee$. These will be concave for a cofinal subset $S \subseteq R(\Sigma)$. It follows from the results in [BPS14, Chapter 4] that for $\Sigma' \in S$, the piecewise linear concave function $\psi^\vee_{\Sigma'}$ induces a continuous toric metric on $D_{\Sigma'}$.

Question B.1. With notations as above, if we define the *toric b-metric degree* $\deg_{\mathfrak{X}_\Sigma}(\boldsymbol{D},\psi)$ by

$$\deg_{\mathfrak{X}_\Sigma}(\boldsymbol{D},\psi) := \varprojlim_{\Sigma' \in R(\Sigma)} \deg_{X_{\Sigma'}}\left(D_{\Sigma'}, \psi^\vee_{\Sigma'}\right),$$

is it true that the equation

$$\deg_{\mathfrak{X}_\Sigma}(\boldsymbol{D},\psi) = \int_{\mathrm{int}(\Delta_{\boldsymbol{D}})} \psi\, \mathrm{d}\lambda$$

holds true?

On $\mathrm{Proj}(b\text{-}R(\boldsymbol{D}))$

Let $\boldsymbol{D}$ be a toric b-divisor on X_Σ. Recall the (not necessarily finitely generated) graded algebra

$$b\text{-}R(\boldsymbol{D}) = \bigoplus_{\ell\geq 0} H^0\left(X_\Sigma, \ell\boldsymbol{D}\right)$$

from Definition 1.5.12.

Question B.2. What is $X := \mathrm{Proj}(b\text{-}R(\boldsymbol{D}))$?

At this point we say more about Question B.2. Note that in the same way that we have a net of fans $\{\Sigma'\}_{\Sigma'\in R(\Sigma)}$, we have a net of polytopes $\{P_{D_{\Sigma'}}\}_{\Sigma'\in R(\Sigma)}$, a net of finite dimensional vector spaces $\{H^0(X_{\Sigma'}, D_{\Sigma'})\}_{\Sigma'\in R(\Sigma)}$ and a net of finitely generated graded algebras $\{R(D_{\Sigma'})\}_{\Sigma'\in R(\Sigma)}$. Now, for any $\Sigma' \in R(\Sigma)$ we have an inclusion of graded algebras

$$i_{\Sigma'} \colon b\text{-}R(\boldsymbol{D}) \hookrightarrow R(D_{\Sigma'}),$$

inducing a rational morphism of schemes $\beta_{\Sigma'}$ as depicted in Figure B.1. Here, $U_{\Sigma'}$ denotes the domain of definition of $\beta_{\Sigma'}$.

$$\begin{tikzcd}
\mathrm{Proj}\left(R\left(D_{\Sigma'}\right)\right) \arrow[r, dashed, "\beta_{\Sigma'}"] & X \\
U_{\Sigma'} \arrow[u, hook] \arrow[ur, "p_{\Sigma'}"'] &
\end{tikzcd}$$

Figure B.1: The rational morphism $\beta_{\Sigma'}$

If $\boldsymbol{D}$ is Cartier, then we have that $b\text{-}R(\boldsymbol{D}) = R(D_{\Sigma'})$ for some $\Sigma' \in R(\Sigma)$. Since $R(D_{\Sigma'})$ is finitely generated, its Proj scheme is projective. Moreover, it may be even toric. Later we will give some precise combinatorial conditions which guarantee that this happens. A more difficult question is: what is X when $\boldsymbol{D}$ is not Cartier? In [Rus96] the author addresses the question whether $\mathrm{Proj}(R(D))$ is a quasi-projective variety, where D is a big and nef divisor on a complete normal variety such that $R(D)$ is not necessarily finitely generated. Here, the main result is the following ([Rus96, Theorem 1]):

Theorem B.3. *Let Z be a normal complete variety and let D be a big and nef divisor on Z. Then the following conditions are equivalent:*

(1) $\mathrm{Proj}(R(D))$ *is a quasi-projective variety.*

(2) In the following diagram, p_D is proper.

$$\begin{tikzcd}
Z \arrow[r, dashed, "\beta_D"] & \mathrm{Proj}\left(R\left(D\right)\right) \\
U_D \arrow[u, hook] \arrow[ur, "p_D"'] &
\end{tikzcd}$$

Here, U_D is the set of definition of β_D.

We can generalize the above theorem to the b-setting.

Theorem B.4. *Let $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ be a toric big and nef b-divisor. Then, with notations as before, the following conditions are equivalent:*

(1) X is a quasi-projective variety.

(2) $p_{\Sigma'}\colon U_{\Sigma'} \to X$ is a proper morphism for some $\Sigma' \in R(\Sigma)$.

Proof. First, let us recall some standard notation and some basic facts about the Proj construction (see e.g. [Har00, II.2]). Let S be a graded ring. Then $\mathrm{Proj}(S)$ is the set of all homogeneous ideals I which do not contain the irrelevant ideal S_+. For any homogeneous $f \in S_+$ the set $D_+(f)$ is defined by

$$D_+(f) := \{I \in \mathrm{Proj}(S)\colon f \notin I\}.$$

We have an isomorphism of locally ringed spaces $(D_+(f), \mathcal{O}|_{D_+(f)}) \simeq \mathrm{Spec} S_{(f)}$. Also, recall that a variety V is said to be *quasi-affine* if the canonical morphism $V \to \mathrm{Spec}(\Gamma(V, \mathcal{O}_V))$ is proper. We proceed with the proof of the theorem.

(2) implies (1) follows exactly as in [Rus96, Proof of Theorem 1]. To see the other implication, assume that X is a quasi-projective variety. Let f be an element in the irrelevant ideal b-$R(\boldsymbol{D})_+$ of the graded ring b-$R(\boldsymbol{D})$. For any $\Sigma' \in R(\Sigma)$ define

$$U_{\Sigma',f} := \{u \in U_{\Sigma'}\colon f(u) \neq 0\}.$$

Note that

$$U_\Sigma = \bigcup_{f \in R(D)_+} U_{\Sigma,f}.$$

Then the restriction $p_{\Sigma'}|_{U_{\Sigma',f}}\colon U_{\Sigma',f} \to \boldsymbol{D}_+(f)$ corresponds to the identity on b-$R(\boldsymbol{D})_{(f)}$ (which we can identify with $\Gamma\left(U_{\Sigma',f}, \mathcal{O}_{U_{\Sigma,f}}\right)$. Now, the map $p_{\Sigma'}$ is proper if and only if $U_{\Sigma',f}$ is a semi-affine variety for all $f \in b$-$R(\boldsymbol{D})_+$. Let us fix an $f \in b$-$R(\boldsymbol{D})_+$. Since X is a variety, $\boldsymbol{D}_+(f)$ is an affine variety and hence

$$\Gamma\left(\boldsymbol{D}_+(f), \mathcal{O}_{\boldsymbol{D}_+(f)}\right) = b\text{-}R(\boldsymbol{D})_{(f)} = \Gamma\left(U_{\Sigma',f}, \mathcal{O}_{U_{\Sigma',f}}\right)$$

is a finitely generated k-algebra. By [Rus96, Corollary 1] and [Rus96, Proposition 4] in order to prove that $U_{\Sigma',f}$ is semi-affine, it suffices to show that the valuation ring R_ν of every divisorial valuation ν of the function field

$$k(U_{\Sigma',f}) = k(U_{\Sigma'}) = k(X_{\Sigma'}) = k(X_\Sigma)$$

containing $\Gamma(U_{\Sigma',f}, \mathcal{O}_{U_{\Sigma',f}})$ has a center in $U_{\Sigma',f}$. Recall that a divisorial valuation is a discrete valuation of the form $\nu_E = t \cdot \mathrm{ord}_E$ where $t \in \mathbb{R}^*_+$ and E is a prime Weil divisor on some birational model over $X_{\Sigma'}$. Now, let us assume that the valuation $\nu = \nu_E$ has no center in $U_{\Sigma',f}$. We may assume that E is a prime toric divisor. Let $\Sigma_\nu \in R(\Sigma)$ be the minimal fan such that $E \subseteq X_{\Sigma_\nu}$ is a prime divisor. Now, the valuation ν having no center in $U_{\Sigma',f}$ means precisely that E is one of the prime divisors appearing in $(b\text{-}\mathrm{div}(f))_{\Sigma_\nu} \sim D_{\Sigma_\nu}$. By [Rus96, Propostion 2] there exists a $\varphi \in R_\nu = \mathcal{O}_{X_{\Sigma_\nu}, x_i}$ with $\varphi \notin \Gamma(U_{\Sigma_\nu,f})$.

We have shown the following: if $f \in b$-$R(\boldsymbol{D})_+$, then for every divisorial valuation ν on $k(X_\Sigma)$ there exists a fan $\Sigma_\nu \in R(\Sigma)$ and a regular function $\varphi_\nu \in \Gamma(U_{\Sigma_\nu,f})$ such that $\varphi_\nu \notin R_\nu$. Now, the statement of the theorem follows by taking $\tilde{\Sigma}$ to be the minimal fan over all divisorial valuations. Indeed, we have $\varphi_\nu \notin R_\nu$ and $\varphi_\nu \in \Gamma(U_{\Sigma_\nu,f}) \subseteq \Gamma(U_{\tilde{\Sigma},f})$. Hence R_ν does not contain $\Gamma(U_{\tilde{\Sigma},f})$, concluding the proof of the theorem. ∎

Remark B.5. The above theorem tells us that in general, in the non-polyhedral case, X is not going to be a quasi-projective variety. Indeed, the open sets $U_{\Sigma'}$ consist of blow-ups of toric varieties with points removed. These are in general not semi-affine varieties.

We conclude this section with the following remark.

Remark B.6. Consider the morphism

$$c_{\Sigma'}\colon X_{\Sigma'} \to \operatorname{Proj}(R(D_{\Sigma'}))$$

induced by the inclusion of algebras

$$i_{\Sigma'}\colon b\text{-}R(\boldsymbol{D}) \hookrightarrow R(D_{\Sigma'}).$$

The morphism $c_{\Sigma'}$ contracts all curves C in $X_{\Sigma'}$ such that $C \cdot D_{\Sigma'} = 0$. Moreover, bigness of $D_{\Sigma'}$ implies that $c_{\Sigma'}$ is a birational morphism (see [Laz04b]). We now give more details about the properties of $c_{\Sigma'}$.

About the contraction morphism $c_{\Sigma'}$

We describe properties of the the contraction morphism

$$c_{\Sigma'}\colon X_{\Sigma'} \to \operatorname{Proj}(R(D_{\Sigma'})).$$

More precisely, we will describe the cases in which the latter is a toric variety and if thus is the case, we describe the situation in which $c_{\Sigma'}$ is a toric morphism. We refer to [Wis02] for proofs and definitions.

It turns out that we can associate a (possibly degenerate) fan $\tilde{\Sigma}'$ to the projective variety $\operatorname{Proj}(R(D_{\Sigma'}))$. We will see that $\operatorname{Proj}(R(D_{\Sigma'}))$ is a toric variety if and only if the fan $\tilde{\Sigma}'$ is non-degenerate. Before describing the fan $\tilde{\Sigma}'$, let us state some basic facts about the intersection theory on toric varieties.

Definition B.7. Let $\operatorname{NE}(X_\Sigma) \subseteq N_1(X_\Sigma)$ the cone of effective 1-cycles (also called the Mori cone).

The Toric Cone Theorem gives us a very nice description of this cone.

Theorem B.8. *(Toric Cone Theorem) Let X_Σ be a complete toric variety. We have*

$$\operatorname{NE}(X_\Sigma) = \sum_{\sigma\in\Sigma(n-1)} \mathbb{R}_{\geq 0} \cdot [V(\sigma)]$$

where $V(\sigma)$ denotes the toric 1-dimensional subvariety associated to the $(n-1)$-dimensional cone σ and we use brackets $[\cdot]$ to denote the corresponding numerical class. In particular $\operatorname{NE}(X_\Sigma)$ is a closed rational polyhedral cone. It is strictly convex if and only if X is projective.

Proof. See [Wis02]. ■

Now, fix an element σ in $\Sigma(n-1)$. Assume that σ is equal to $\langle e_1,\dots,e_{n-1}\rangle$ where the e_i are primitive lattice elements on rays spanning σ. The cone σ separates the following two n-dimensional cones in Σ:

$$\delta_n = \langle e_1,\dots,e_{n-1},e_n\rangle \text{ and } \delta_{n+1} = \langle e_1,\dots,e_{n-1},e_{n+1}\rangle \tag{B.1}$$

where e_n and e_{n+1} are primitive rays on opposite sides of σ. We denote by $\delta(\sigma)$ the Minkowski sum $\delta_n + \delta_{n+1}$. We have the following result concerning intersection numbers (see [Wis02]).

Proposition B.9. *Let $D_\tau \subseteq X_\Sigma$ be a divisor corresponding to a ray $\tau = \mathbb{R}_{\geq 0} \cdot v_\tau \in \Sigma(1)$, with $v_\tau \in N$ primitive. Then $D_\tau \cdot V(\sigma) = 0$ if and only if $v_\tau \notin \{e_1, e_2, \ldots, e_n, e_{n+1}\}$.*

Now, we can finally describe our fan $\tilde{\Sigma}'$ from Σ'. Let $D_{\Sigma'} = \sum_{\tau_i \in \Sigma'(1)} d_{\tau_i} D_{\tau_i}$. Then for every $\sigma \in \Sigma'(n-1)$ such that (using the above notation) $v_{\tau_i} \notin \{e_1, e_2, \ldots, e_n, e_{n+1}\}$ for all τ_i such that $d_{\tau_i} \neq 0$, we remove σ from Σ' and replace the two adjacent cones δ_n and δ_{n+1} in $\Sigma'(n)$ with the cone $\delta(\sigma)$. Then, doing this for every such σ and taking respectively their faces, we get a complete (possibly degenerate) fan $\tilde{\Sigma}'$. We have the following proposition.

Proposition B.10. *Let $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ be a toric b-divisor. For any $\Sigma' \in R(\Sigma)$, let*

$$c_{\Sigma'} \colon X_{\Sigma'} \to \operatorname{Proj}\left(R\left(D_{\Sigma'}\right)\right)$$

be the birational morphism given above. Then, if the fan $\tilde{\Sigma}'$ described above is non-degenerate, we have that

$$\operatorname{Proj}\left(R\left(D_{\Sigma'}\right)\right) = X_{\tilde{\Sigma}'}$$

is a toric variety and $c_{\Sigma'}$ is the toric morphism induced by the naturally defined morphism

$$\Sigma' \to \tilde{\Sigma}'$$

given by the assignment

$$\sigma \mapsto \begin{cases} \delta(\sigma), & \text{if } \sigma \in \Sigma'(n-1) \text{ gets removed,} \\ \sigma, & \text{otherwise.} \end{cases}$$

Proof. This follows from the Cone Theorem B.8 and the construction of $\tilde{\Sigma}'$. ∎

Remark B.11. Note that the non-degeneracy of the fan $\tilde{\Sigma}'$ depends on which cones $\sigma \in \Sigma'(n-1)$ are removed. There is a precise necessary condition on the combinatorics of these cones which ensures that this happens (see [Wis02, Toric Contraction Theorem I]).

Now, if we impose some conditions on the b-divisor, we can say a bit more about how the fans $\tilde{\Sigma}'$ behave as we move in $R(\Sigma)$. We have the following proposition.

Proposition B.12. *Let $\boldsymbol{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ be a toric b-divisor. Suppose that for $\Sigma' \in R(\Sigma)$ one has $\operatorname{supp}(D_{\Sigma'}) \cap \{D_\tau \colon \tau \in \Sigma'(1) \setminus \Sigma(1)\} \neq \emptyset$. Then it follows that*

$$\tilde{\Sigma}' = \tilde{\Sigma} \cap \{\rho \in \Sigma'(1) \notin \Sigma(1)\}.$$

Proof. Let $\Sigma' \in R(\Sigma)$ and suppose that a cone $\sigma \in \Sigma'(n-1)$ gets removed. Let $\sigma = \langle e_1, \ldots, e_{n-1} \rangle$ and e_n, e_{n+1} as in (B.1) and let $D_{\Sigma'} = \sum_{\tau_i \in \Sigma'(1)} d_{\tau_i} D_{\tau_i}$. By hypothesis, we have that for all $\tau_i \in \Sigma'(1)$ such that $d_{\tau_i} \neq 0$, $v_{\tau_i} \notin \{e_1, \ldots, e_{n-1}, e_n, e_{n+1}\}$. Let $D_\Sigma = \sum_{\tau_j \in \Sigma(1)} d_{\tau_j} D_{\tau_j}$. By the definition of toric b-divisors, we have that $v_{\tau_j} \notin \{e_1, \ldots, e_{n-1}, e_n, e_{n+1}\}$ for all $\tau_j \in \Sigma(1)$ such that $d_{\tau_j} \neq 0$. We have two cases. First, if $\sigma \in \Sigma(n-1)$, then σ would have been removed in $c_\Sigma \colon X_\Sigma \to \operatorname{Proj}(R(D_\Sigma))$. Hence, if $\sigma \in \Sigma(n-1)$, we have that $\delta(\sigma) \in \tilde{\Sigma}'$ which implies that $\delta(\sigma) \in \tilde{\Sigma}$. Otherwise, if $\sigma \notin \Sigma(n-1)$, then this means that σ comes from a subdivision of one of the cones in Σ. Hence, one of the e_i's is already one of the v_{τ_i}'s. This cannot happen. ∎

Appendix C

3-dimensional example of computing correction terms

In this appendix, we give 3-dimensional examples where we compute the correction terms given in Section 3.5 in the polyhedral case.

Example C.1. We compute the correction term of one incarnation of the tropically nef b-divisor given by the concave, conical function which is $\frac{xy+xz+yz}{x+y+z}$ on the positive orthant (note that we are in the polyhedral setting). Consider the fan in $\Sigma \subseteq \mathbb{R}^3$ depicted in Figure C.1. Here, the dots "$\cdots$" mean that there are still more (not drawn) cones making the fan indeed smooth. However, note that since they do not lie in the star of some cone in the positive orthant, they don't contribute to the local correction terms so we don't bother to draw them.

Now, let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be toric b-divisors on X_Σ given by

$$\begin{aligned}\boldsymbol{D}_1 =& -D_{111} - \frac{1}{2}D_{101} - \frac{1}{2}D_{011} - \frac{1}{2}D_{110} + \frac{1}{2}D_{1\text{-}11} + \frac{1}{2}D_{1\text{-}111} + \frac{1}{2}D_{11\text{-}1}\\ &+ D_{\text{-}211} + D_{1\text{-}21} + D_{11\text{-}2} + 2D_{\text{-}1\text{-}1\text{-}1} + \dots,\\ \boldsymbol{D}_2 =& \frac{1}{2}D_{1\text{-}11} + \frac{1}{2}D_{1\text{-}111} + \frac{1}{2}D_{11\text{-}1} + D_{\text{-}211} + D_{1\text{-}21} + D_{11\text{-}2} + 2D_{\text{-}1\text{-}1\text{-}1} + \dots,\end{aligned}$$

where, in order to simplify notation, we have denote by D_{v_i} the toric divisor corresponding to the ray $\tau_{v_i} = \mathbb{R}_{\geq 0}v_i$. Also, abusing notation, we will denote by ϕ_1 and by ϕ_2 instead of by $\phi_{\boldsymbol{D}_1}$ and by $\phi_{\boldsymbol{D}_2}$, respectively, the corresponding piecewise linear, concave functions. Let K_1 and K_2 be the polytopes corresponding to the stability sets of ϕ_1 and ϕ_2, respectively. We clearly have $K_1 \subseteq K_2$.

Our aim now is to compute the correction term $\boldsymbol{D}_2^3 - \boldsymbol{D}_1^3$ and to express it as a sum of local correction terms associated to faces of Δ_2. We proceed by steps.

- **Full correction term $\boldsymbol{D}_2^3 - \boldsymbol{D}_1^3$.**
  Using that all the cones that contribute to the computation are smooth and that in this case we have

$$D_{v_i}D_{v_j}D_{v_k} = \begin{cases} 0 & \text{if } v_i, v_j, v_k \text{ are not rays of a cone } \sigma \in \Sigma,\\ 1 & \text{otherwise.}\end{cases}$$

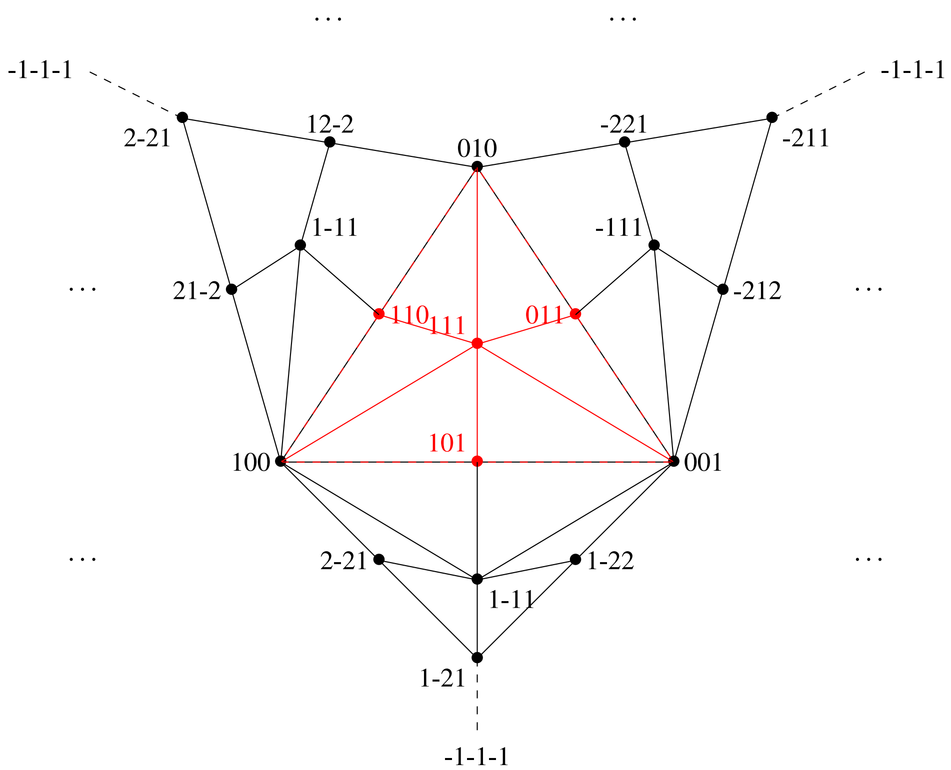


Figure C.1: 2-dimensional depiction of the fan Σ for $v_i \neq v_j \neq v_k$. Hence, we get

$$\begin{aligned} \boldsymbol{D}_2^3 - \boldsymbol{D}_1^3 = & D_{111}^3 + \frac{1}{8}D_{101}^3 + \frac{1}{8}D_{011}^3 + \frac{1}{8}D_{110}^3 + \frac{3}{2}D_{111}^2 D_{101} + \frac{3}{2}D_{111}^2 D_{011} + \frac{3}{2}D_{111}^2 D_{110} \\ & + \frac{3}{4}D_{111}D_{101}^2 + \frac{3}{4}D_{111}D_{011}^2 + \frac{3}{4}D_{111}D_{110}^2 + \frac{3}{8}D_{1\text{-}11}^2 D_{101} + \frac{3}{8}D_{\text{-}111}^2 D_{011} \\ & + \frac{3}{8}D_{11\text{-}1}^2 D_{110} - \frac{3}{8}D_{1\text{-}11}D_{101}^2 - \frac{3}{8}D_{\text{-}111}D_{011}^2 - \frac{3}{8}D_{11\text{-}1}D_{110}^2. \end{aligned} \tag{C.1}$$

Now, to compute these intersection numbers we use the following fact, which is stated as an exercise in [Ful93, Section 5.1].
Fact 1: Suppose that an $(n-1)$-dimensional cone σ is the common face of two nonsingular n-dimensional cones γ' and γ''. Let $u_1,\dots,u_{n-1}$ be the minimal lattice points on the edges of σ, and let v' and v'' be the minimal lattice points on the other edges of γ' and γ'' respectively. Then there are unique integers $a_1,\dots,a_{n-1}$ such that

$$v' + v'' = a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}$$

and one can compute the intersection numbers by the formula

$$D_kV(\sigma) = D_1D_2\cdots D_k^2\cdots D_{n-1} = -a_k.$$

One can generalize this formula to the simplicial case. In this case we get unique *rational* numbers a_i.

Now, for computing the top self-intersection numbers we use the following result which can be found in e.g. [EH16].

Fact 2: Let X be a smooth threefold and let $C \subseteq X$ be a smooth curve of genus g. Let X' be the blowup of X along C with exceptional divisor E. Then the top intersection number of E in X' is given by $E^3 = -\deg N_{C|X}$, where $\deg N_{C|X} = 2g - 2 - K_XC$.

Fact 2 has the following interpretation in the toric setting.

Fact 2 (toric): Let Σ be a 3-dimensional fan and let $\sigma \in \Sigma$ be a two-dimensional cone. Let Σ' be the fan corresponding to the blow up of Σ along σ and let E_τ be the exceptional divisor corresponding to a ray $\tau \in \Sigma'(1)$. Then the top intersection number of E_τ is given by

$$E_\tau^3 = K_\Sigma V(\sigma) = \left(\sum_{\tau\in\Sigma(1)} -D_\tau\right) V(\sigma).$$

Hence, the top intersection number E_τ^3 can be computed using Fact 1, as can be seen in the following figure.

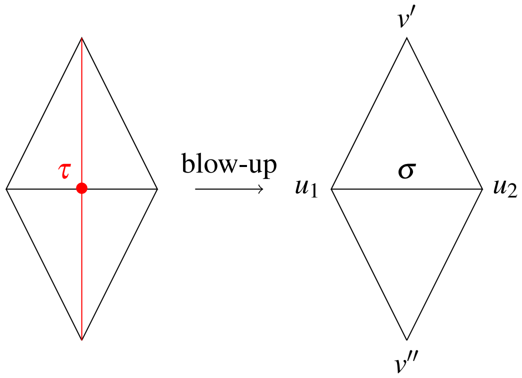


Figure C.2: Local picture for toric intersection numbers

Returning to our example and using the above facts, we compute

$$\begin{aligned}
&D_{111}^3 = 1\\
&D_{101}^3 = D_{110}^3 = D_{011}^2 = 2\\
&D_{111}^2 D_{101} = D_{111}^2 D_{011} = D_{111}^2 D_{110} = 0\\
&D_{111} D_{101}^2 = D_{111} D_{011}^2 = D_{111} D_{110}^2 = -1\\
&D_{1\text{-}11}^2 D_{101} = D_{\text{-}111}^2 D_{011} = D_{11\text{-}1}^2 D_{110} = 0\\
&D_{1\text{-}11} D_{101}^2 = D_{\text{-}111} D_{011}^2 = D_{11\text{-}1} D_{110}^2 = -1.
\end{aligned}$$

Hence, inserting these values in Formula (C.1), we obtain

$$\begin{aligned}
\boldsymbol{D}_2^3 - \boldsymbol{D}_1^3 &= 1 + \frac{1}{8}\cdot 3\cdot 2 + \frac{3}{2}\cdot 3\cdot 0 + \frac{3}{4}\cdot 3\cdot(-1) + \frac{3}{8}\cdot 3\cdot 0 - \frac{3}{8}\cdot 3\cdot(-1)\\
&= \frac{7}{4} - \frac{9}{4} + \frac{9}{8} = \frac{5}{8}
\end{aligned}$$

for the full correction term.

- **Local correction terms** c_F.
  The only faces of Δ_2 which give a non-trivial correction term are the three 1-dimensional faces, whose corresponding 2-dimensional cones in Σ_2 are

$$\{\sigma_{ij} = \langle e_i, e_j\rangle : i, j \in \{1,2,3\}, i \neq j\},$$

together with the vertex corresponding to the 3-dimensional cone

$$\sigma_{123} = \langle e_1, e_2, e_3\rangle.$$

To simplify notation, we will denote these correction terms by c_{ij} for $i, j \in \{1,2,3\}$, $i \neq j$, and by c_{123}, respectively. Note that the c_{ij}'s correspond to the correction terms coming from the blow ups of the three torus invariant curves, while the c_{123} corresponds to the correction term coming from the blow up at the torus fixed point. Now, by symmetry, for the corrections coming from curves, it suffices to compute c_{13}. By Theorem 3.5.27 we have

$$\begin{aligned}
c_{13} &= \sum_{\substack{\tau\in\mathrm{relint}(\sigma_{13})\\ \tau\in\Sigma(1)}} \left((\phi_1(v_\tau)-\phi_2(v_\tau)) \left(\sum_{i=0}^{2} (\phi_1^\tau)^{n-1-i}(\phi_2^\tau)^i \right) \cdot [\Sigma(\tau)] \right)\\
&= \sum_{\substack{\tau\in\mathrm{relint}(\sigma_{13})\\ \tau\in\Sigma(1)}} \left((\phi_1(v_\tau)-\phi_2(v_\tau)) \left(\sum_{i=0}^{2} \phi_1^{n-1-i}\phi_2^i \right) D_\tau \right)\\
&= \frac{1}{2}\left(\phi_1^2 D_{101}+\phi_1\phi_2 D_{101}+\phi_2^2 D_{101}\right)\\
&= \frac{1}{8}D_{101}^3+\frac{1}{2}D_{111}^2 D_{101}+\frac{1}{2}D_{111}D_{101}^2+\frac{3}{8}D_{1\text{-}11}^2 D_{101}-\frac{3}{8}D_{1\text{-}11}D_{101}^2\\
&= \frac{1}{4}-\frac{1}{2}+\frac{3}{8}=\frac{1}{8}.
\end{aligned}$$

By symmetry, we get

$$\begin{aligned}
c_{12} &= \frac{1}{8}D_{110}^3+\frac{1}{2}D_{111}^2 D_{110}+\frac{1}{2}D_{111}D_{110}^2+\frac{3}{8}D_{11\text{-}1}^2 D_{110}-\frac{3}{8}D_{11\text{-}1}D_{110}^2=\frac{1}{8},\\
c_{23} &= \frac{1}{8}D_{011}^3+\frac{1}{2}D_{111}^2 D_{011}+\frac{1}{2}D_{111}D_{011}^2+\frac{3}{8}D_{\text{-}111}^2 D_{011}-\frac{3}{8}D_{\text{-}111}D_{011}^2=\frac{1}{8}.
\end{aligned}$$

Now, for the correction term coming from the point, we compute

$$\begin{aligned}
c_{123} &= 1\left(\phi_1^2 D_{111}+\phi_1\phi_2 D_{111}+\phi_1\phi_2 D_{111}+\phi_2^2 D_{111}\right)\\
&= D_{111}^3+D_{111}^2\left(D_{101}+D_{011}+D_{110}\right)+\frac{1}{4}D_{111}\left(D_{101}^2+D_{011}^2+D_{110}^2\right)\\
&= 1-\frac{3}{4}=\frac{1}{4}.
\end{aligned}$$

Note that

$$3\cdot\frac{1}{8}+\frac{1}{4}=\frac{5}{8}.$$

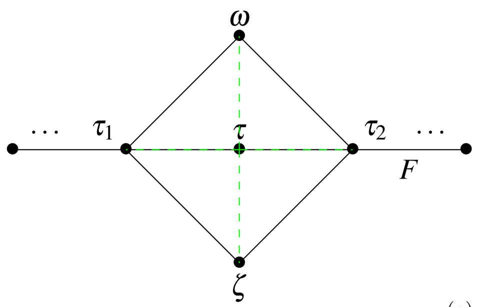
Example C.2. In the 3-dimensional case, we give a general formula for the correction terms in the polyhedral case. From the calculations carried out in the previous example one can expect that in the polyhedral case the local correction terms c_F depend only on the local combinatorical picture around the cone σ_F, i.e. on the values of ϕ_i at the rays of its star, and on the top intersection numbers E_τ^3 for $\tau\in\mathrm{relint}(\sigma_F)$. Let us be more precise. We start with two 3-dimensional rational polytopes

$$K_1\subseteq K_2.$$

Let $\Sigma=\Sigma_{K_2\setminus K_1}$ be a difference fan. Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be its corresponding nef toric b-divisors on X_Σ. We start by computing the correction term coming from the blow up of a toric curve. Let $F\leq K_2$ be a 1-dimensional face and let σ_F be the corresponding two dimensional cone. As before, in order to simplify notation, we write ϕ_1 and ϕ_2 for the piecewise linear, concave functions corresponding to $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, respectively. Again, by Theorem 3.5.27, we have the formula

$$\begin{aligned}
c_F &= \sum_{\substack{\tau\in\mathrm{relint}(\sigma_F)\\ \tau\in\Sigma(1)}} \left((\phi_1(v_\tau)-\phi_2(v_\tau) \left(\sum_{i=0}^{2} (\phi_1(\tau))^{n-1-i}(\phi_2(\tau))^i \right) \cdot [\Sigma(\tau)] \right)\\
&= \sum_{\substack{\tau\in\mathrm{relint}(\sigma_F)\\ \tau\in\Sigma(1)}} (\phi_1(v_\tau)-\phi_2(v_\tau)\left(\phi_1^2 D_\tau+\phi_1\phi_2 D_\tau+\phi_2^2 D_\tau\right).
\end{aligned}$$

Our aim is to give a formula for the factor on the right in terms of the quantities mentioned earlier. For each $\tau \in \mathrm{relint}(\sigma_F)$ we have a local picture as in Figure C.3 with $\tau = \tau_1 + \tau_2$. Let ω and ζ be rays and a_1, b_1, a_2, b_2, c and d integer numbers as in the following figure.



$$\omega + \zeta = a_1\tau + b_1\tau_1$$

$$\omega + \zeta = a_2\tau + b_2\tau_2$$

$$\omega + \zeta = c\tau_1 + d\tau_2$$

$$\overset{(\star)}{\Longrightarrow} c = a_2\ ,\ d = a_1 \text{ and } b_1 = a_2 - a_1 = \text{-}b_2.$$

Figure C.3: Local picture for the correction term c_F

We assume that $\phi_2|_{\sigma_F} = 0$ and that $\phi_1(v_\zeta) = \phi_2(v_\zeta)$. Using Facts 1 and 2 from the previous example, the intersection numbers $\phi_1^2 D_\tau$ and $\phi_1\phi_2 D_\tau$ are given by

$$\begin{aligned}
\phi_1^2 D_\tau &= \left(-\phi_1(v_\omega)D_\omega - \phi_1(v_\zeta)D_\zeta - \phi_1(v_{\tau_1})D_{\tau_1} - \phi_1(v_{\tau_2})D_{\tau_2} - \psi_1(v_\tau)D_\tau\right)^2 D_\tau \\
&= \phi_1(v_\tau)^2 D_\tau^3 - \phi_1(v_{\tau_1})^2 b_1 - 2\phi_1(v_\tau)\phi_1(v_{\tau_2})a_2 - \phi_1(v_{\tau_2})^2 b_2 - 2\phi_1(v_\tau)\phi_1(v_{\tau_1})a_1 \\
&\quad - 2\phi_1(v_\tau)\phi_1(v_\omega) \quad + 2\phi_1(v_{\tau_1})\phi_1(v_\omega) + 2\phi_1(v_{\tau_2})\phi_1(v_\omega) - 2\phi_1(v_\tau)\phi_1(v_\zeta) \\
&\quad + 2\phi_1(v_{\tau_1})\phi_1(v_\zeta) + 2\phi_1(v_{\tau_2})\phi_1(v_\zeta).
\end{aligned}$$

$$\begin{aligned}
\phi_1\phi_2 D_\tau &= \left(-\phi_1(v_\omega)D_\omega - \phi_1(v_\zeta)D_\zeta - \phi_1(v_{\tau_1})D_{\tau_1} - \phi_1(v_{\tau_2})D_{\tau_2} - \phi_1(v_\tau)D_\tau\right)\left(-\phi_2(v_\zeta)D_\zeta\right) D_\tau \\
&= \phi_2(v_\zeta)\left(\phi_1(v_{\tau_1}) + \phi_1(v_{\tau_2}) + \phi_1(v_\tau)\right) \\
&= \phi_1(v_\zeta)\left(\phi_1(v_{\tau_1}) + \phi_1(v_{\tau_2}) - \phi_1(v_\tau)\right). \\
\phi_2^2 D_\tau &= \phi_2(v_\zeta)^2 D_\zeta^2 D_\tau \\
&= 0.
\end{aligned}$$

Putting everything together, we get

$$\begin{aligned}
c_F &= \sum_{\substack{\tau \in \mathrm{relint}(\sigma_F) \\ \tau \in \Sigma(1)}} \left(\phi_1(v_\tau) - \phi_2(v_\tau)\right)\left(\phi_1^2 D_\tau + \phi_1\phi_2 D_\tau + \phi_2^2 D_\tau\right) \\
&= \sum_{\substack{\tau \in \mathrm{relint}(\sigma_F) \\ \tau \in \Sigma(1)}} \phi_1(v_\tau)\Big[\phi_1(v_\tau)^2 D_\tau^3 - \phi_1(v_{\tau_1})\left(2\phi_1(v_\tau)a_1 + \phi_1(v_{\tau_1})b_1\right) - \phi_1(v_{\tau_2})\left(2\phi_1(v_\tau)a_2 + \phi_1(v_{\tau_2})b_2\right) \\
&\qquad + 2\phi_1(v_\omega)\left(\phi_1(v_{\tau_1}) + \phi_1(v_{\tau_2}) - \phi_1(v_\tau)\right) + 3\phi_1(v_\zeta)\left(\phi_1(v_{\tau_1}) + \phi_1(v_{\tau_2}) - \phi_1(v_\tau)\right)\Big].
\end{aligned}$$

Furthermore, one has $D_\tau^3 = (c + d - 2)$. Then using $(\star)$ and reordering the terms, we have that the correction term c_F is given by

$$c_F = \sum_{\substack{\tau \in \mathrm{relint}(\sigma_F) \\ \tau \in \Sigma(1)}} \Bigg[\phi_1(v_\tau)^3 [a_1 + a_2 - 2] + \phi_1(v_\tau)^2 \left[-2a_1\phi_1(v_{\tau_1}) - 2a_2\phi_1(v_{\tau_2}) - 2\phi_1(v_\omega) - 3\phi_1(v_\zeta)\right]$$
$$+ \phi_1(v_\tau)\big[a_1\phi_1(v_{\tau_1})^2 - a_2\phi_1(v_{\tau_1})^2 - a_1\phi_1(v_{\tau_2})^2 + a_2\phi_1(v_{\tau_2})^2 +$$
$$2\phi_1(v_{\tau_1})\phi_1(v_\omega) + 2\phi_1(v_{\tau_2})\phi_1(v_\omega) + 3\phi_1(v_{\tau_1})\phi_1(v_\zeta) + 3\phi_1(v_{\tau_2})\phi_1(v_\zeta)\big]\Bigg]. \tag{C.2}$$

Note that this confirms the calculation made above in the previous example. Indeed, there we had $\tau = \mathbb{R}_{\geq 0}(101)$, $\tau_1 = \mathbb{R}_{\geq 0}(100)$, $\tau_2 = \mathbb{R}_{\geq 0}(001)$, $\zeta = \mathbb{R}_{\geq 0}(1\text{-}11)$ and $\omega = \mathbb{R}_{\geq 0}(111)$. Also, $\phi_1(v_\tau) = \frac{1}{2}$, $\phi(v_\omega) = 1$, $\phi_1(v_\zeta) = \phi_2(v_\zeta) = \frac{-1}{2}$, $\phi_1(v_{\tau_1}) = \phi_1(v_{\tau_2}) = \phi_2(v_{\tau_1}) = \phi_2(v_{\tau_2}) = 0$ and hence $a_1 = a_2 = 2$.

Thus, Equation (C.2) reads

$$c_{13} = \frac{1}{8}(2) + \frac{1}{4}\left(-2 + \frac{3}{2}\right) + \frac{1}{2}(0) = \frac{1}{8}.$$

List of Symbols

Algebra

k	algebraically closed field of characteristic 0
$\mathbb{T}$	algebraic torus over k
χ^m	character of a torus $\mathbb{T}$ for an element m in its lattice of characters
$R(D)$	graded ring of global sections of multiples of a divisor D in an algebraic variety X
$\mathrm{Cox}(X)$	Cox ring of an algebraic variety X
b-$R(\boldsymbol{D})$	graded ring of global sections of multiples of a nef toric b-divisor $\boldsymbol{D}$ on a toric variety X_Σ
$\mathrm{Cox}\left(\mathfrak{X}_\Sigma\right)$	b-Cox ring of a toric variety X_Σ
$A_{\boldsymbol{D}}$	graded algebra of almost integral type associated to a toric b-divisor $\boldsymbol{D}$
$S(A)$	semigroup attached to an algebra of almost integral type A
H_S	Hilbert function of a semigroup S
H_A	Hilbert function of a graded algebra A
$N^1(X)$	$\mathbb{R}$-linear space of numerical classes of Cartier divisors on a complete variety X
$N^1(\mathcal{X}_\Sigma)$	$\mathbb{R}$-linear space of numerical classes of toric b-divisors on a toric variety X_Σ
$\Delta(X)$	global Okounkov body of a smooth projective variety X
$\Delta\left(\mathcal{X}_\Sigma\right)$	global Okounkov b-body of a toric variety X_Σ
$\mathrm{Big}\left(\mathcal{X}_\Sigma\right)$	set of classes of big toric b-divisors
$\mathrm{Nef}\left(\mathcal{X}_\Sigma\right)$	set of classes of nef toric b-divisors

Arithmetic

$\mathbb{H}$	complex upper half plane
$\Gamma(N)$	principal congruence subgroup of level N
$J_{k,m}\left(\Gamma(N)\right)$	space of Jacobi forms of weight k and index m for $\Gamma(N)$

$J^{\text{cusp}}_{k,m}(\Gamma(N))$	space of Jacobi cusp forms of weight k and index m for $\Gamma(N)$
$J^{\text{weak}}_{k,m}(\Gamma(N))$	space of weak Jacobi forms of weight k and index m for $\Gamma(N)$
$\overline{L}_{k,m,N}$	hermitian line bundle of Jacobi forms of weight k and index m for $\Gamma(N)$
$\boldsymbol{D}_{k,m,N}$	b-divisor of Jacobi forms of weight k and index m for $\Gamma(N)$
$\boldsymbol{\mathcal{L}}_{k,m,N}$	b-line bundle of Jacobi forms of weight k and index m for $\Gamma(N)$

Combinatorics

N	lattice of rank n
N_R	$N \otimes_{\mathbb{Z}} R$ where R is any commutative ring with unity
N^{prim}	set of primitive elements in the lattice N
v	element in $N_{\mathbb{R}}$
Σ	fan in $N_{\mathbb{R}}$
σ, τ, γ	cones in Σ
$\tau \leq \sigma$	τ is a face of σ
$\sigma^{\vee}$	dual cone of a cone σ
$\text{relint}(\sigma)$	relative interior of a cone σ
$N(\sigma)$	quotient lattice with respect to σ: $N/(N \cap \mathbb{R}\sigma)$
N_σ	lattice spanned by σ: $N \cap \mathbb{R}\sigma$
$\Sigma(\sigma)$	star of Σ at σ in $N(\sigma)_{\mathbb{R}}$
$\Sigma(k)$	set of k-dimensional cones in Σ
v_τ	primitive vector in N spanning a ray $\tau \subseteq N_{\mathbb{R}}$
τ_v	ray in $N_{\mathbb{R}}$ spanned by a vector v in N
Σ_P	normal fan of a polyhedron P
Σ_I	subfan of a fan Σ spanned by a subset of rays $I \subseteq \Sigma(1)$
$\epsilon(\Sigma)$	Euler characteristic of a fan Σ: $\sum_{d=0}^{n}(-1)^d \#\Sigma(d)$
$R(\Sigma)$	set of smooth subdivisions of Σ
Σ', Σ''	elements in $R(\Sigma)$
$\Sigma'' \geq \Sigma'$	Σ'' is a smooth subdivision of Σ'
v_σ	barycenter of a smooth cone σ

$\Sigma^*(v_\sigma)$	barycentric subdivision of Σ corresponding to a barycenter v_σ of a smooth cone $\sigma \in \Sigma$
$\tilde{\phi}_{\boldsymbol{D}}$	rational, conical, $\mathbb{Q}$-valued function on N^{prim} corresponding to a toric b-divisor $\boldsymbol{D}$
$\phi_{\boldsymbol{D}}$	rational, conical, $\mathbb{R}$-valued, concave function on $N_{\mathbb{R}}$ corresponding to a nef toric b-divisor $\boldsymbol{D}$
ψ_D	virtual support function corresponding to a toric Cartier divisor D
$f^\vee$	Legendre–Fenchel dual of a concave function $f\colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$
v_α, v_β	primitive vectors in $\mathbb{Z}^2$ defined by a minimal regular subdivision of Σ_{P_v} for a primitive vector $v \in \mathbb{Z}^2$
$\mu_{\boldsymbol{D}}$	jumping function on N^{prim} associated to a toric b-divisor $\boldsymbol{D}$ on a toric surface
M	the dual lattice of N: $N^\vee = \operatorname{Hom}(N, \mathbb{Z})$
M_R	$M \otimes_{\mathbb{Z}} R$ where R is any commutative ring with unity
m	element in $M_{\mathbb{R}}$
P	rational polyhedron in $M_{\mathbb{R}}$
$Q \leq P$	Q is a face of P
F	facet of a polytope
P_D	rational polytope in $M_{\mathbb{R}}$ associated to a toric divisor D in X_Σ
Δ_f	stability set in $M_{\mathbb{R}}$ of a concave function $f\colon N_{\mathbb{R}} \to \underline{\mathbb{R}}$
$\Delta_{\boldsymbol{D}}$	bounded convex set in $M_{\mathbb{R}}$ associated to a (not necessarily nef) toric b-divisor $\boldsymbol{D}$
$\mathcal{K}_n$	set of n-dimensional bounded, compact convex sets in $\mathbb{R}^n$
K	bounded, compact, convex set in $M_{\mathbb{R}}$ or in $\mathbb{R}^n$
h_K	support function of a bounded, compact, convex set K in $M_{\mathbb{R}}$
f_K	Ehrhart function of a bounded, compact, convex set K in $M_{\mathbb{R}}$
P_v	Newton polyhedron associated to a primitive vector $v \in N^2$
$\langle m, v \rangle$	pairing of $m \in M$ or $M_{\mathbb{R}}$ with $v \in N$ or $N_{\mathbb{R}}$
Δ_A	Okounkov body attached to a graded algebra of almost integral type A
Π	weakly embedded conical complex
σ^α	rational polyhedral cone of the conical complex Π
M^α	lattice of real-valued continuous functions on σ^α
N^α	dual lattice of M^α

ι_Π	weak embedding of the conical complex Π
N^Π	lattice corresponding to the weak embedding of the complex Π
M^Π	dual lattice of N^Σ
$\langle m, v\rangle_\alpha$	pairing of $m \in M^\alpha$ or $M^\alpha_\mathbb{R}$ with $v \in N^\alpha$ or $N^\alpha_\mathbb{R}$, respectively
σ, τ	cones in Π
N^Π_σ	the space $N^\sigma \cap \mathrm{Span}(\iota_\Sigma(\sigma))$ for a cone σ in Π
$\Pi(k)$	set of k-dimensional cones in Π
$\Pi(\tau)$	star of a conical polyhedral complex Π at a cone τ
$R(\Pi)$	directed set of all smooth subdivisions of Π
$v_{\sigma/\tau}$	lattice normal vector of σ relative to τ
$M_k(\Pi)$	group of k-dimensional Minkowski weights on Π
c	Minkowski weight on Π
$\hat{c}$	normalized Minkowski weight on Π
$Z_k(\Pi)$	group of tropical k-cycles on Π
$[c]$	class of a Minkowski weight c in $Z_k(\Pi)$
$\mathrm{Div}(\Pi)$	group of Cartier divisors on Π
ϕ	Cartier divisor on Π
$\mathrm{cpDiv}(\Pi)$	group of combinatorially principal Cartier divisors on Π
$\mathrm{cpCl}(\Pi)$	group of combinatorially principal Cartier divisors on Π modulo linear equivalence
$\phi \cdot [c]$	tropical intersection product between a Cartier divisor and a tropical cycle on Π
$\phi \hat{\odot} [c]$	normalized tropical intersection product between a Cartier divisor and a tropical cycle on Π
$[\Pi]$	map assigning the value 1 to every top-dimensional cone in Π
$\phi_1 \cdots \phi_n$	top intersection number of Cartier divisors on Π
$(\phi_{\Pi'})_{\Pi' \in R(\Pi)}$	b-divisor on Π
Υ	convex decomposition of a collection of convex sets in $M_\mathbb{R}$
$\partial(f)$	sup-differential of a concave function f
$\Upsilon(f)$	convex decomposition of $\mathrm{dom}(\partial f)$ associated to a concave function f
$\mathcal{L}f$	Legendre–Fenchel correspondence associated to a concave function f

Σ_K	(possibly infinite) complete fan in $N_{\mathbb{R}}$ corresponding to a convex set K given by the Legendre–Fenchel duality
$F_1 \sim F_2$	F_1 is related to F_2 where F_1 and F_2 are exposed faces of convex sets K_1 and K_2 with $K_1 \subseteq K_2$ respctively
K_F	correction set in $K_2 \setminus K_1$ associated to a face $F \leq K_2$ with $K_1 \subseteq K_2$
c_F	correction term corresponding to a multiple of the volume of the correction set K_F
$\mathrm{cone}(v_1,\dots,v_k)$	cone spanned by a set of vectors $\{v_1,\dots,v_k\}$
$\mathrm{cone}(S)$	cone over a semigroup S
$\mathrm{convhull}(\cdot)$	convex hull of points

Geometry

X_Σ	n-dimensional toric variety associated to a polyhedral fan Σ
U_σ	open affine subset of X_Σ corresponding to a cone $\sigma \in \Sigma$: $\mathrm{Spec}\,(k[\sigma^\vee \cap M])$
x_σ	distinguished closed point in X_Σ with respect to a cone $\sigma \in \Sigma$
$O(\sigma)$	torus orbit of the distinguished closed point x_σ in X_Σ for a cone $\sigma \in \Sigma$
$V(\sigma)$	toric subvariety associated to a cone $\sigma \in \Sigma$: $\overline{O(\sigma)} = X_{\Sigma(\sigma)}$
π	toric morphism between toric varieties $X_{\Sigma_1} \to X_{\Sigma_2}$ induced by a lattice morphism $\pi\colon N_1 \to N_2$ compatible with fans $\Sigma_1 \subseteq N_{1\mathbb{R}}$ and $\Sigma_2 \subseteq N_{2\mathbb{R}}$
$\pi_{\Sigma'}$	proper birational toric morphism $X_{\Sigma'} \to X_\Sigma$ induced by a subdivision Σ' of Σ
$\mathbb{T}\text{-}\mathrm{Ca}(X_\Sigma)_{\mathbb{Q}}$	set of toric Cartier $\mathbb{Q}$-divisors on X_Σ
$\mathbb{T}\text{-}\mathrm{We}(X_\Sigma)_{\mathbb{Q}}$	set of toric Weil $\mathbb{Q}$-divisors on X_Σ
D_Σ	toric Weil divisor on X_Σ: $\sum_{\tau\in\Sigma(1)} a_\tau D_\tau$
D_P	toric Weil divisor associated to a rational poytope P
D_ψ	toric Cartier divisor associated to a virtual support function ψ
$\mathfrak{X}_\Sigma$	toric Riemann–Zariski space
$\mathbb{T}\text{-}\mathrm{Ca}(\mathfrak{X}_\Sigma)_{\mathbb{Q}}$	set of toric Cartier b-divisors on X_Σ
$\mathbb{T}\text{-}\mathrm{We}(\mathfrak{X}_\Sigma)_{\mathbb{Q}}$	set of toric Weil b-divisors on X_Σ
$\boldsymbol{D}$	toric Weil b-divisor : $(D_{\Sigma'})_{\Sigma'\in R(\Sigma)}$
$\boldsymbol{E}$	toric Cartier b-divisor : $(E_{\Sigma'})_{\Sigma'\in R(\Sigma)}$
$\boldsymbol{D}^n$	degree of a toric b-divisor $\boldsymbol{D}$
$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$	mixed degree of a collection of toric b-divisors $\boldsymbol{D}_1,\dots,\boldsymbol{D}_n$

$H^0(X_\Sigma, \boldsymbol{D})$	space of global sections of a toric b-divisor $\boldsymbol{D}$
$h^0(X_\Sigma, \boldsymbol{D})$	dimension of $H^0(X_\Sigma, \boldsymbol{D})$
$\mathrm{Cl}(X)$	class group of an algebraic variety X
$\mathrm{Pic}(X)$	Picard group of an algebraic variety X
$\mathrm{Cl}(\mathfrak{X}_\Sigma)$	b-class group of a toric variety X_Σ
$\mathrm{Pic}(\mathfrak{X}_\Sigma)$	b-Picard group of a toric variety X_Σ
μ_ϕ	discrete measure or limit measure associated to a $\mathbb{Q}$-Cartier divisor ϕ on Π or to a continuous tropically nef b-divisor ϕ on Π, respectively
$\mu_{\phi_1,\ldots,\phi_{n-1}}$	mixed measure or mixed limit measure associated to a collection of $\mathbb{Q}$-Cartier divisors or of continuous tropically nef b-divisors on Π, respectively
$\mathcal{H}^k$	k-dimensional Hausdorff measure on $\mathbb{R}^n$
$S_{n-1}(K,\cdot)$	surface area measure associated to a convex set K in $\mathcal{K}_n$
$S(K_1,\ldots,K_n,\cdot)$	mixed surface area measure associated to a collection of convex sets $K_1,\ldots,K_n$ in $\mathcal{K}_n$
$U \hookrightarrow X_\Pi$	toroidal embedding with associated weakly embedded conical polyhedral complex Π
S_J	stratum of a toroidal embedding for some index set J
$\overline{S}_J$	closure of a stratum of a toroidal embedding for some index set J
$\mathrm{Star}(S_J)$	combinatorial open set corresponding to a stratum S_J
M^{S_J}	lattice corresponding to a stratum S_J
N^{S_J}	dual lattice corresponding to a stratum S_J
σ^{S_J}	cone corresponding to a stratum S_J
Π_X	rational conical polyhedral complex associated to a toroidal embedding $U \hookrightarrow X$
S^σ	stratum corresponding to a cone σ in Π_X
$\overline{S^\sigma}$	closure of a stratum corresponding to a cone σ in Π_X
D_τ	irreducible boundary component of a toroidal embedding $U \hookrightarrow X$ associated to a ray $\tau \in \Pi_X(1)$
Δ_X	Clemens complex associated to a pair (X,D) consisting of an algebraic variety X and a snc divisor D on X
M^{Π_X}	lattice of invertible functions on U modulo constants corresponding to a toroidal embedding $U \hookrightarrow X$
N^{Π_X}	dual lattice of M^{Π_X}

ι_{Π_X}	weak embedding of the conical complex Π_X
$\phi_{\mathcal{F}}$	function associated to a toroidal fractional ideal sheaf $\mathcal{F}$
$\mathrm{Div}_0(X)_{\mathbb{Q}}$	group of toroidal $\mathbb{Q}$-Cartier divisors on a toroidal embedding $U \hookrightarrow X$
ϕ_F	$\mathbb{Q}$-Cartier divisor on Π_X corresponding to a toroidal divisor F
$\mathrm{cpDiv}_0(X)_{\mathbb{Q}}$	subgroup of cp toroidal divisors on a toroidal embedding $U \hookrightarrow X$
$\mathfrak{X}_\Pi$	toroidal Riemann–Zariski space of a toroidal embedding $U \hookrightarrow X_\Pi$
$\mathrm{We}\,(\mathfrak{X}_\Pi)_{\mathbb{Q}}$	group of toroidal Weil b-divisors on a toroidal embedding $U \hookrightarrow X$
$\boldsymbol{D}$	toroidal b-divisor
$\mathrm{trop}(C)$	tropicalization of an algebraic cycle C on a toroidal embedding $U \hookrightarrow X$
$\boldsymbol{D}^n$	degree of a toroidal b-divisor $\boldsymbol{D}$
$\boldsymbol{D}_1 \cdots \boldsymbol{D}_n$	mixed degree of a collection of toroidal b-divisors $\boldsymbol{D}_1, \dots, \boldsymbol{D}_n$
$X^0(N)$	modular curve of level N
$X(N)$	compactified modular curve of level N by adding the cusps
$E^0(N)$	universal elliptic curve of level N
$E(N)$	smooth toroidal compactification of the universal elliptic curve of level N lying over $X(N)$
$\mathrm{Pic}\,(\mathfrak{X}_{\Pi_X})$	Picard group of the toroidal Riemann–Zariski space $\mathfrak{X}_{\Pi_X}$

Operators

vol	volume operator on $M_{\mathbb{R}}$ computed with respect to the Haar measure on $M_{\mathbb{R}}$ normalized so that the lattice M has covolume 1
vol_k	k-dimensional volume operator on $\mathbb{R}^n$
vol_L	lattice volume operator on $M_{\mathbb{R}}$
MV	mixed volume of a collection of convex sets in $M_{\mathbb{R}}$
V	mixed volume defined by a multiple of MV
$\deg_{\mathcal{X}_\Sigma}$	degree function on $\mathrm{Big}\,(\mathcal{X}_\Sigma) \cap \mathrm{Nef}\,(\mathcal{X}_\Sigma)$

Other Symbols

$\underline{\mathbb{R}}$	the extended real line $\mathbb{R} \cup \{-\infty\}$

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 29. März 2017 Ana María Botero Carrillo